CSE 105
THEORY OF COMPUTATION

"Winter" 2018

http://cseweb.ucsd.edu/classes/wi18/cse105-ab/

Today's learning goals

Sipser Section 2.1

- Prove the closure of class of CFLs under certain operations
- Identify when a CFG is ambiguous
- Design a CFG generating a given language

Context-free grammar

Sipser Def 2.2, page 102

(V, Σ, R, S)

Variables: finite set of (usually upper case) variables **V**

Terminals: finite set of alphabet symbols **Σ**

Rules/Productions: finite set of allowed transformations **R**

$$A \to u \qquad A \in V, u \in (V \cup \Sigma)^*$$

Start variable: origination of each derivation **S**

The **language generated by a CFG** (V, Σ, R, S) is { w in Σ^* | starting with the Start variable and applying sequence of rules, can derive w on RHS}

Context-free languages

- L (00*)
- L((0U1)*)
- { abba }
- { $a^n b^n$ | n ≥ 0 } → context free but not regular

Ex: { $0^i 1^j$ | j ≥ i ≥ 0 } recognizable by a PDA

PDAs and CFGs are equally expressive

Theorem 2.20: A language is context-free if and only if some nondeterministic PDA recognizes it.

Consequences

- Quick proof that every regular language is context free
- To prove closure of class of CFLs under a given operation, can choose two modes of proof (via CFGs or PDAs) depending on which is easier

Closure under union

If $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ are CFGs and G_1 generates L_1, G_2 generates L_2, how can we combine the grammars so we generate $L_1 \cup L_2$?

A. $G = (V_1 \cup V_2, \Sigma, R_1 \cup R_2, S_1 \cup S_2)$

B. $G = (V_1 \times V_2, \Sigma, R_1 \times R_2, (S_1, S_2))$

C. $G = (V1 \cup V2 \cup \{S0\}, \Sigma, R_1 \cup R_2 \cup \{S0 \rightarrow S1, S0 \rightarrow S2\}, S0)$

D. We might not always be able to: the class of CFG describable languages might not be closed under union.

Closure under concatenation

If $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ are CFGs and G_1 generates L_1, G_2 generates L_2, how can we combine the grammars so we generate $L_1 L_2$? Assume $V_1 \cap V_2 = \emptyset$

$$G = \left(V_1 \cup V_2 \cup \{S_0\}, \Sigma, R_1 \cup R_2 \cup \{S_0 \to S_1 S_2\}, S_0 \right)$$

CFGs in the wild

V = {E}, Σ = {1, +, x, (,)}, R = { **E → E+E | ExE | (E) | 1** }, S=E

Describing well-formed arithmetic expressions

Which of the followings strings is generated by this CFG?

A. E

B. 11

C. 1+1x1

D. ε

E. I don't know.

E → E+E → E+ExE →

1+ExE → 1+1xE → 1+1x1

Derivations and parsing

E → E+E | ExE | (E) | 1

not a left-most derivation.

Lots of derivations for 1+1x1

$$E \Longrightarrow E+E \Longrightarrow E+E\times E \Longrightarrow 1+E\times E \Longrightarrow 1+1\times E \Longrightarrow 1+1\times 1$$

$$E \Longrightarrow E\times E \Longrightarrow E+E\times E \Longrightarrow 1+E\times E \Longrightarrow 1+1\times E \Longrightarrow 1+1\times 1$$

leftmost derivation.

$$E \Longrightarrow E+E \Longrightarrow 1+E \Longrightarrow 1+E\times E \Longrightarrow 1+1\times E \Longrightarrow 1+1\times 1$$

left-most derivation

Derivations and parsing

E → E+E | ExE | (E) | 1

Lots of derivations for 1+1x1

$$E \Longrightarrow E+E \Longrightarrow E+E\times E \Longrightarrow 1+E\times E \Longrightarrow 1+1\times E \Longrightarrow 1+1\times 1$$

$$E \Longrightarrow E\times E \Longrightarrow E+E\times E \Longrightarrow 1+E\times E \Longrightarrow 1+1\times E \Longrightarrow 1+1\times 1$$

$$E \Longrightarrow E+E \Longrightarrow 1+E \Longrightarrow 1+E\times E \Longrightarrow 1+1\times E \Longrightarrow 1+1\times 1$$

Derivations and parsing

E → E+E | ExE | (E) | 1

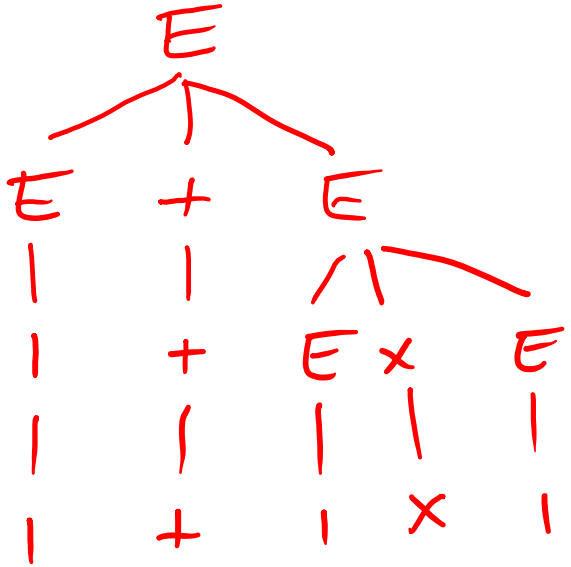
Lots of derivati

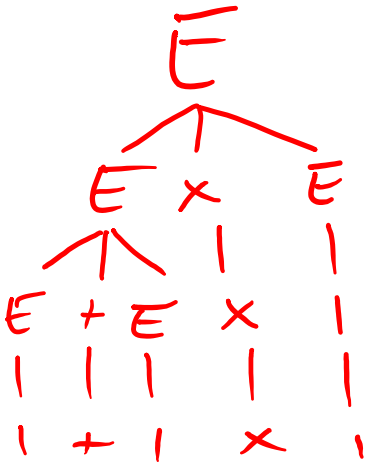
$$E \implies E + E$$

$$E \implies E \times E \qquad + 1 \times 1$$

$$E \implies E + E \implies 1 + E \implies 1 + E \times E \implies 1 + 1 \times E \implies 1 + 1 \times 1$$

A string is **ambiguously derived** in a CFG if it has more than one leftmost derivation i.e. more than one parsing tree


E
E + E
+ E x E
+ x


E
E x E
E + E x
+ x

More complicated language

$\{a^n b^m \mid n \neq m\}$

Exercise: pumping lemma argument.

More complicated language

$$\{a^n b^m \mid n \neq m\}$$

Can rewrite as:

$$\{a^n b^m \mid n > m\} \cup \{a^n b^m \mid n < m\}$$

$G_1 = (\{S_1\}, \{a, b\}, \{S_1 \to a,\ S_1 \to aS_1b,\ S_1 \to aS_1\}, S_1)$

$S_1 \to a \mid aS_1b \mid aS_1$

More complicated language

$$\{a^n b^m \mid n \neq m\}$$

Can rewrite as:

$$\{a^n b^m \mid n > m\} \cup \{a^n b^m \mid n < m\}$$

$G_1 = (\{S_1\}, \{a,b\}, R_1, S_1)$ with rules $S_1 \rightarrow aS_1 b \mid aS_1 \mid a$

$G_2 = (\{S_2\}, \{a,b\}, R_2, S_2)$ with rules $S_2 \rightarrow$ [illegible] $\mid aS_2 \mid a$

$b \mid aS_2 b \mid S_2 b$

$G = (\{S, S_1, S_2\}, \{a,b\}, R, S)$ with rules $S \rightarrow S_1 \mid S_2$, ... $\cup R_1 \cup R_2$

Summary

A language L is

Regular	iff it is described by some regular expression
	iff it is recognized by some DFA
	iff it is recognized by some NFA
Context-free	iff it is recognized by some (nondet) PDA
	iff it is generated by some (possibly ambiguous) CFG

What's left?

- **Fact**: Every regular language is a context-free language.
- **Fact**: There are context-free languages that are nonregular.
- **Fact**: There are countably infinitely many regular languages.
- **Fact**: There are countably infinitely many context-free languages.

Most languages are not context-free languages!

Examples of non-context-free languages

- $\{ a^n b^n c^n \mid 0 \leq n \}$ Sipser Ex 2.36
- $\{ a^i b^j c^k \mid 0 \leq i \leq j \leq k \}$ Sipser Ex 2.37
- $\{ w\, w \mid w \text{ is in } \{0,1\}^* \}$ Sipser Ex 2.38

To prove… Pumping lemma for CFLs (won't cover in CSE 105)

Closure properties of …

The class of regular languages is closed under

- Union
- Concatenation
- Star
- Complementation
- Intersection
- Difference
- Reversal

The class of context-free languages is closed under

- Union
- Concatenation
- Star
- Reversal

The class of context-free languages is not closed under

- Intersection
- Complement
- Difference

For next time

IndivHW4 due Tuesday, February 13

For Monday, pre-class reading: pp. 166-167