

CSE 20

DISCRETE MATH

Winter 2017

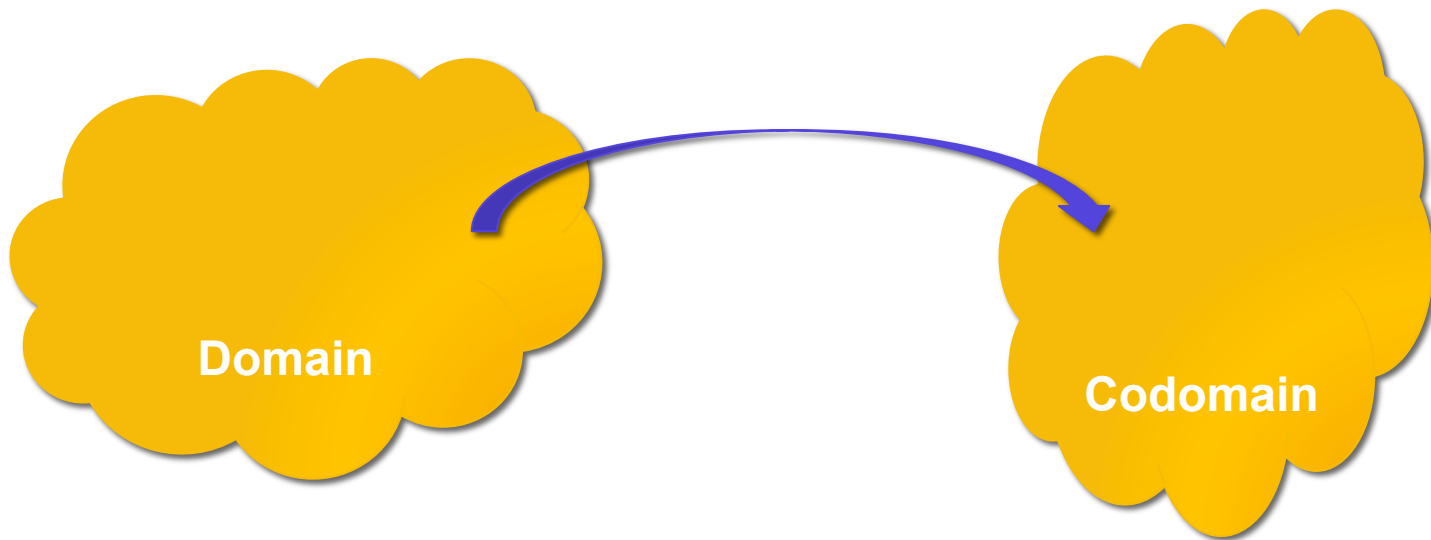
<http://cseweb.ucsd.edu/classes/wi17/cse20-ab/>

Today's learning goals

- Represent functions in multiple ways
- Define and prove properties of: domain of a function, image of a function, composition of functions
- Determine and prove whether a function is one-to-one, onto, bijective
- Apply the definition and properties of floor function, ceiling function, factorial function
- Define and compute the cardinality of a set: Finite sets, countable sets, uncountable sets
- Use functions to compare the sizes of sets

Functions

Rosen Sec 2.3; p. 138



Function

Mapping

Transformation

$$\forall a(a \in D \rightarrow \exists! b(b \in C \wedge f(a) = b))$$

To specify a function

(1) Domain

(2) Codomain

(3) Assignment

Operations on functions

Rosen p. 141,147

If $f: A \rightarrow \mathbf{R}$, $g: A \rightarrow \mathbf{R}$

$f+g: A \rightarrow \mathbf{R}$

$fg: A \rightarrow \mathbf{R}$

If $f: B \rightarrow C$, $g: A \rightarrow B$

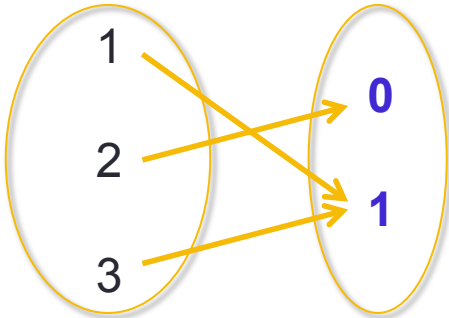
$f \circ g: A \rightarrow C$

Properties of functions

Rosen p. 143

possible

- A function f is **onto** means **at least one input for every output** (surjective)

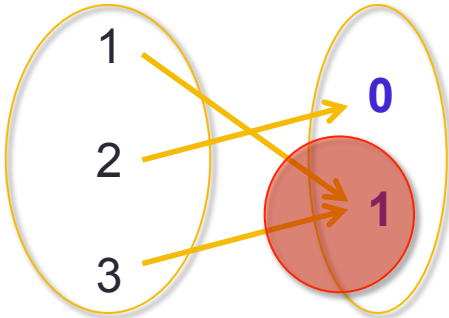


$$\forall b(b \in C \rightarrow \exists a(a \in D \wedge f(a) = b))$$

Properties of functions

Rosen p. 141

- A function f is **one-to-one** means **no duplicate images** (injective)



How can we formalize this?

A. $\forall a \forall b ((a \in D \wedge b \in D) \rightarrow f(a) \neq f(b))$

B. $\forall a \forall b ((a \in D \wedge b \in D) \rightarrow (f(a) = f(b) \rightarrow a = b))$

C. $\forall a \forall b ((a \in C \wedge b \in C) \rightarrow a \neq b)$

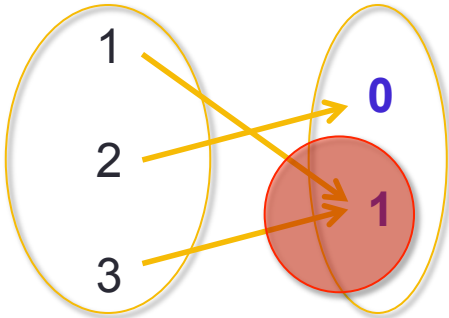
D. $\forall a \forall b ((a \in C \wedge b \in C) \rightarrow f(a) \neq f(b))$

E. None of the above

Properties of functions

Rosen p. 141

- A function f is **one-to-one** means **no duplicate images** (injective)

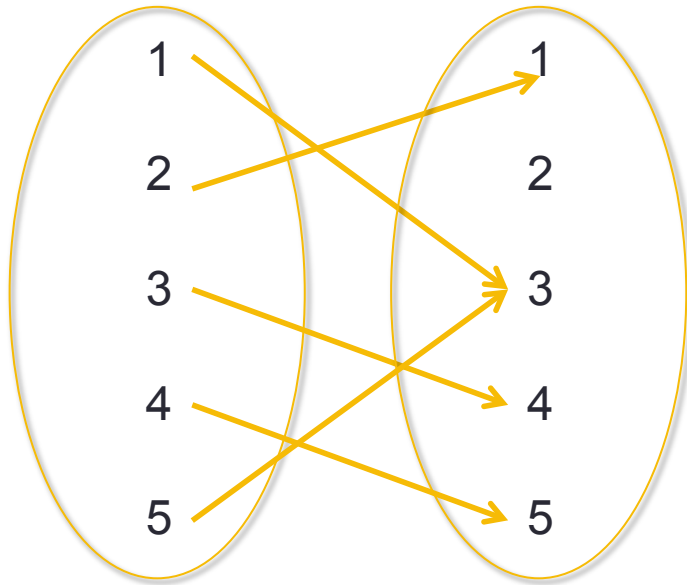


$$\forall a \forall b ((a \in D \wedge b \in D) \rightarrow (f(a) = f(b) \rightarrow a = b))$$

$$\forall a \forall b ((a \in D \wedge b \in D) \rightarrow (a \neq b \rightarrow f(a) \neq f(b)))$$

Onto? One-to-one?

Consider the function over domain and codomain $\{1,2,3,4,5\}$ defined by



This function is

- A. Well defined, onto, and one-to-one.
- B. Well defined, but neither onto nor one-to-one.
- C. Well defined, onto, but not one-to-one.
- D. Not well-defined, not onto, not one-to-one.
- E. None of the above.

Onto? One-to-one?

Consider the function over domain and codomain $\mathbf{R}^{\geq 0}$ defined by

$$f(x) = x^2$$

This function is

- A. Well defined, onto, and one-to-one.
- B. Well defined, but neither onto nor one-to-one.
- C. Well defined, onto, but not one-to-one.
- D. Not well-defined, not onto, not one-to-one.
- E. None of the above.

Proving a function is ...

Rosen p. 145

Define $f:\{0,1\}^* \rightarrow \mathbf{N}$ by $f(w) = |w|$, or formally, f is defined recursively by

$$\begin{cases} f(\lambda) = 0 \\ f(w0) = f(w) + 1 \\ f(w1) = f(w) + 1 \end{cases}$$

Fact: This function is onto.

Proving a function is ...

Define $f: \{0,1\}^* \rightarrow \mathbf{N}$ by $f(w) = |w|$, or formally, f is defined recursively by

$$\begin{cases} f(\lambda) = 0 \\ f(w0) = f(w) + 1 \\ f(w1) = f(w) + 1 \end{cases}$$

Fact: This function is not one-to-one.

Proving a function is ...

Let $A = \{1,2,3\}$ and $B = \{2,4,6\}$.

Define a function from the power set of A to the power set of B by:

$$f : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

$$f(X) = X \cap B$$

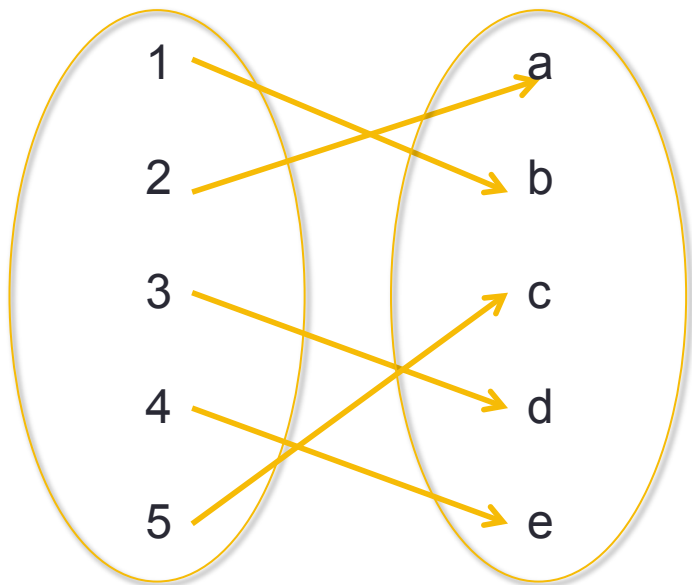
Well-defined?

Onto?

One-to-one?

One-to-one + onto

Rosen p. 144



one-to-one correspondence

bijection

invertible

The **inverse** of a function $f: A \rightarrow B$ is the function $g: B \rightarrow A$ such that

$$\forall b(b \in B \rightarrow (g(b) = a \leftrightarrow f(a) = b))$$

One-to-one + onto

Rosen p. 144

What's the inverse function of

$$f(x) = x^3 + 1 ?$$

A. $g(x) = x^{1/3} - 1$

B. $g(x) = (x - 1)^{1/3}$

C. $g(x) = \frac{1}{x^3 + 1}$

D. More than one of the above.

E. None of the above.

one-to-one correspondence

bijection

invertible

The **inverse** of a function $f: A \rightarrow B$ is the function $g: B \rightarrow A$ such that

$$\forall b(b \in B \rightarrow (g(b) = a \leftrightarrow f(a) = b))$$

Functions and subsets

Rosen Theorem 2, p 174

One-to-one: $\forall a \forall b ((a \in D \wedge b \in D) \rightarrow (f(a) = f(b) \rightarrow a = b))$

Onto: $\forall b (b \in C \rightarrow \exists a (a \in D \wedge f(a) = b))$

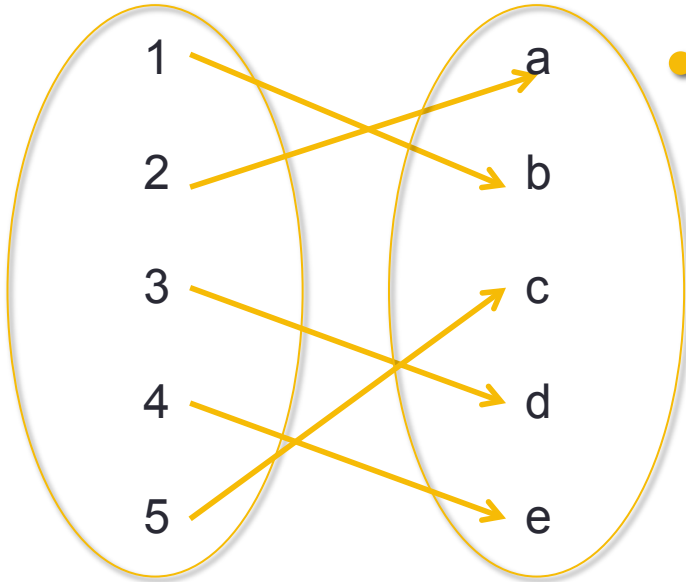
Bijection: both one-to-one and onto

Which of the following is true?

- A. If A is a subset of B then there is a one-to-one function from A to B
- B. If A is a subset of B then there is an onto function from A to B
- C. If A is a subset of B then there is a bijection from A to B
- D. None of the above.
- E. I don't know

One-to-one + onto

Rosen p. 144



Fact: for finite sets A and B, there is a bijection between them if and only if $|A| = |B|$.

Beyond finite sets

Rosen Section 2.5

For all sets, we say

$|A| = |B|$ if and only if there is a bijection between them.

Which of the following is true?

A. $|\mathbf{Z}| = |\mathbf{N}|$

B. $|\mathbf{N}| = |\mathbf{Z}^+|$

C. $|\mathbf{Z}| = |\{0,1\}^*|$

D. All of the above.

E. None of the above.

Sizes and subsets

Rosen Theorem 2, p 174

For all sets A, B we say

$|A| \leq |B|$ if there is a one-to-one function from A to B .

$|A| \geq |B|$ if there is an onto function from A to B .

Cantor-Schroder-Bernstein Theorem: $|A| = |B|$ iff $|A| \leq |B|$ and $|A| \geq |B|$

Which of the following is true?

A. If A is a subset of B then $|A| \leq |B|$

B. If A is a subset of B then $|A| \neq |B|$

C. If A is a subset of B then $|A| = |B|$

D. None of the above.

E. I don't know

Beyond finite sets

Rosen Section 2.5

For all sets, we say

$|A| = |B|$ if and only if there is a bijection between them.

Which of the following is true?

A. $|\mathbf{Q}| = |\mathbf{Q}^+|$

B. $|\mathbf{Q}^+| = |\mathbf{N} \times \mathbf{N}|$

C. $|\mathbf{N}| = |\mathbf{Q}|$

D. All of the above.

E. None of the above.

Cardinality

Rosen Defn 3 p. 171

- Finite sets
- Countably infinite sets
- Uncountable sets

$|A| = n$ for some nonnegative int n

$|A| = |\mathbf{Z}^+|$ (informally, can be listed out)

Infinite but not in bijection with \mathbf{Z}^+