CSE 20
DISCRETE MATH

Winter 2017

http://cseweb.ucsd.edu/classes/wi17/cse20-ab/

Today's learning goals

- Explain the steps in a proof by (strong) mathematical induction
- Use (strong) mathematical induction to prove
  - correctness of identities and inequalities
  - properties of algorithms
  - properties of geometric constructions
- Represent functions in multiple ways
- Use functions to define sequences: arithmetic progressions, geometric progressions
- Use and prove properties of recursively defined functions and recurrence relations (using induction)
- Use and interpret Sigma notation

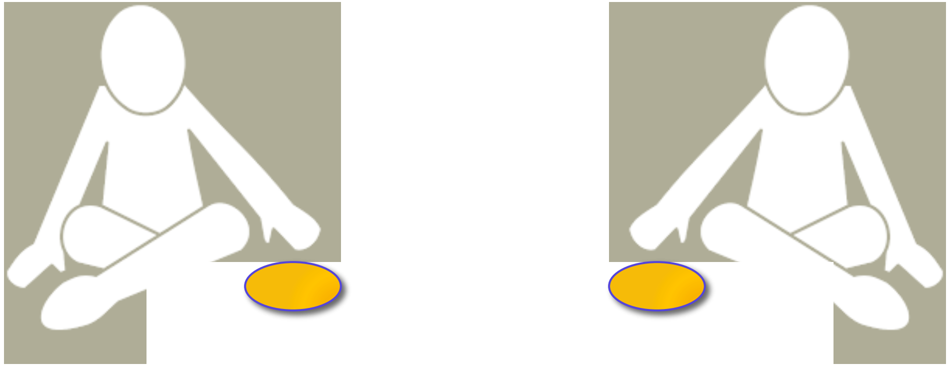
Nim

Two players take turns removing
any positive # of jellybeans
at a time from one of two piles in front of them. **The piles start out with equal #s.**
The player who removes the **last jellybean wins** the game.

A. The first player has a strategy to always win.
B. The second player has a strategy to always win.
C. One of the players has a strategy to always win, but which player depends on how many jellybeans there are.
D. Who wins is random.
E. None of the above.

Nim

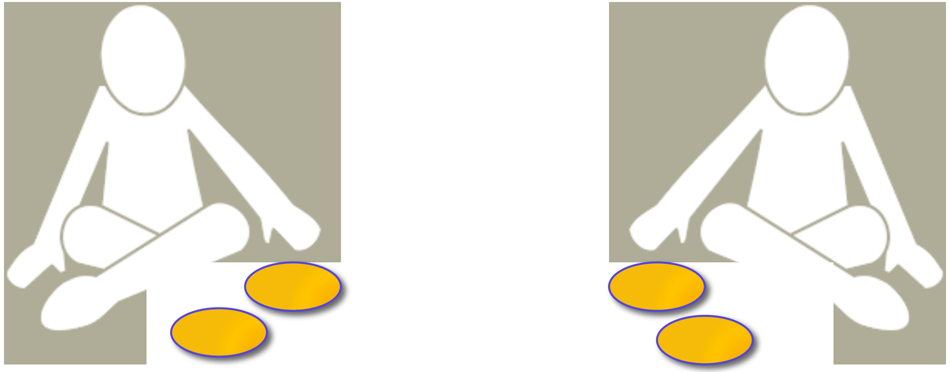
Two players take turns removing **any positive # of jellybeans** at a time from one of two piles in front of them. **The piles start out with equal #s.** The player who removes the **last jellybean wins** the game.



n=1 *Who wins?*

Nim

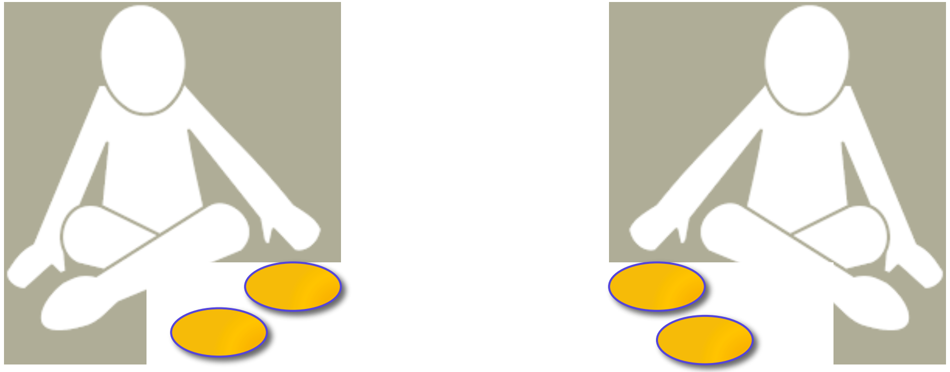
Two players take turns removing **any positive # of jellybeans** at a time from one of two piles in front of them. **The piles start out with equal #s.** The player who removes the **last jellybean wins** the game.



n=2 *Who wins?*

Nim

Two players take turns removing **any positive # of jellybeans** at a time from one of two piles in front of them. **The piles start out with equal #s.** The player who removes the **last jellybean wins** the game.



Idea: 2^{nd} player takes the same amount 1^{st} player took but from opposite pile.

*…Game reduces to **same setup** but with fewer jellybeans.*

Strong induction

Rosen p. 334

To show that some statement P(n) is true about all positive integers n,

1. Verify that P(1) is true.
2. Let k be an arbitrary positive integer. Show that

$$[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$$

is true.

Strong induction hypothesis

Nim

Two players take turns removing **any positive # of jellybeans** at a time from one of two piles in front of them. **The piles start out with equal #s.** The player who removes the **last jellybean wins** the game.

Theorem: the second player can always guarantee a win.

Proof: By Strong Mathematical Induction, on # jellybeans in each pile.

1. Basis step WTS if piles each have 1, then 2^{nd} player can win.
2. Strong Induction hypothesis Let k be a positive integer. Assume that 2^{nd} player can win whenever there are j jellybeans in each pile, for each j between 1 and k (inclusive).
3. Induction step WTS 2^{nd} player has winning strategy when start with k+1 jellybeans in each pile.

Fibonacci numbers

Rosen p. 158, 347

$$f_0 = 1,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$$

What are some sample values?

How quickly do these value grow?

Fibonacci numbers

Rosen p. 158, 347

$$f_0 = 1,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$$

Theorem: For each integer n>=2, $f_n \geq 1.5^{n-2}$

Proof: By Strong Mathematical Induction, on n>=2.

1. Basis step WTS $f_2 \geq 1.5^{2-2}$.
2. Strong Induction hypothesis Let k be an integer, k>=2. Assume inequality is true for each **integer j, 2<=j<=k.**
3. Induction step WTS statement is true about f_{k+1}.

Fibonacci numbers

Rosen p. 158, 347

$$f_0 = 1,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$$

1. Basis step WTS $f_2 >= 1.5^{2-2}$.

LHS = f_2 = 1 + 1 = 2.

RHS = $1.5^{2-2} = 1.5^0 = 1$.

Since 2>1, LHS>RHS so, in particular, LHS>= RHS ☺

Fibonacci numbers

Rosen p. 158, 347

$$f_0 = 1, \; f_1 = 1, \; f_n = f_{n-1} + f_{n-2}$$

Induction step Let k be an integer with k>=2.

Assume as the **strong induction hypothesis** that

$$f_j >= 1.5^{j-2}$$

for each integer j with 2<=j<=k.

WTS that $\mathbf{f_{k+1} >= 1.5^{(k+1)-2}}$

By definition of Fibonacci numbers, since k+1 > 1, $f_{k+1} = f_k+f_{k-1}$.

Therefore, LHS = $f_{k+1} = f_k+f_{k-1}$.

Idea: apply strong induction hypothesis to k and k-1. Can we do it?

Fibonacci numbers

Rosen p. 158, 347

…

Case 1: k=2 and WTS that $\mathbf{f_3 >= 1.5^{(3)-2}}$

Case 2: k>2 and WTS that $\mathbf{f_{k+1} >= 1.5^{(k+1)-2}}$ and strong IH applies to k, k-1 (because both k, k-1 are greater than or equal to 2 and less than k+1).

So let's prove each of these cases in turn:

Case 1: k=2 and WTS that $\mathbf{f_3 >= 1.5^{(3)-2}}$

By definition of Fibonacci numbers, LHS = $f_3=f_2+f_1=2+1=3$.

By algebra, RHS = $1.5^{3-2}=1.5$ Since 3>1.5, LHS>RHS ☺

Fibonacci numbers

Rosen p. 158, 347

…

Case 2: k>2 and WTS that $\mathbf{f_{k+1} >= 1.5^{(k+1)-2}}$ and strong IH applies to k, k-1 (because both k, k-1 are greater than or equal to 2 and less than k+1).

LHS = $f_{k+1} = f_k + f_{k-1} >= 1.5^{k-2} + 1.5^{(k-1)-2} = 1.5^{k-3}(1.5+1) = 1.5^{k-3}(2.5)$

$> 1.5^{k-3}(2.25) = 1.5^{k-3}1.5^2 = 1.5^{k-1} = 1.5^{(k+1)-2}$ = RHS.

Def of Fibonacci numbers

Strong induction hypothesis

Flavors of induction

- Mathematical induction
- Structural induction
- Strong induction

Fulfilling promises

- We now have all the tools we need to rigorously prove
  - Correctness of **greedy change-making algorithm** with quarters, dimes, nickels, and pennies *Proof by contradiction, Rosen p. 199*
  - The **division algorithm** is correct *Strong induction, Rosen p. 341*
  - **Russian peasant multiplication** is correct *Induction*
  - Largest **n-bit binary** number is 2^n-1 *Induction, Rosen p. 318*
  - Correctness of **base b conversion** (Algorithm 1 of 4.2), *Strong induction*
  - Size of the **power set** of a finite set with n elements is 2^n *Induction, Rosen p. 323*
  - Any int greater than 1 can be written as **product of primes** *Strong induction, Rosen p. 323*
  - There are infinitely many **primes** *Proof by contradiction, Rosen p. 260*
  - **Sum** of geometric progressions $\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1}$ when $r \neq 1$, *Induction, Rosen p. 318*

Cautionary tales

- The **basis step** is absolutely necessary … and might need more than one!
- Make sure to stay in the **domain**.

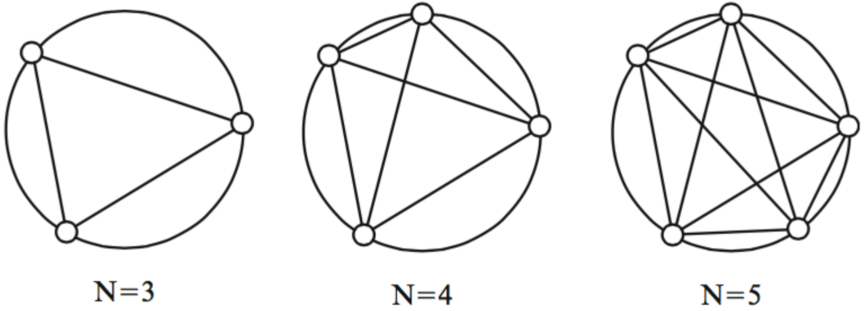
Recommended practice

Section 5.1 #49, 50, 51

Section 5.2 #32

- A few **examples** do not guarantee a pattern:

cake cutting conundrum. Join all pairs of points among N marked on circumference of cake.



Reminders

- Exam 2 is next class: **Tuesday Feb 28**
  - One note card allowed.
  - Seat map posted on Piazza.

- Review sessions this weekend.
  - Sat, Feb 25, 11:00am – 12:50pm PETERSON 108
  - Mon, February 27, 8:00pm – 9:50pm PETERSON 110

- Extra office hours available.
- Practice exam available on Piazza.