
TOPICS Sets and Induction.

READING Rosen Sections 1.8, 2.1, 2.2, 5.1, 5.2.

KEY CONCEPTS Proof strategies, counterexample, (constructive) example, direct proof, contrapositive proof, proof by contradiction, sets, element, subset, empty set, power set, Cartesian product, finite set, union, intersection, set difference, cardinality, mathematical induction.

1. (10 points)

- (a) Prove that for every integer n , if 3 divides n^2 then 3 divides n .
- (b) Prove that $\sqrt{3}$ is irrational. *Hint: you may use the fact that fractions can be written in lowest terms, and you can use the result you proved in part (a).*

2. (8 points) In this question, you'll explore power sets. Before you start, review the definition from class and from the textbook (Definition 6 on page 120).

- (a) Let A be a set, and consider the statement

$$(\{x\} \in \mathcal{P}(A)) \rightarrow (x \in A)$$

The next paragraph claims to prove this statement. Is this proof correct? Explain why or why not.

Let A be a set and suppose x is an element of A . Then, by definition of the power set, there will be a singleton set containing x in $\mathcal{P}(A)$, that is, $\{x\} \in \mathcal{P}(A)$. Therefore, $\{x\} \in \mathcal{P}(A) \rightarrow x \in A$.

- (b) Prove or disprove the following statement.

$$\text{For all sets } A, B, (\mathcal{P}(A) = \mathcal{P}(B)) \leftrightarrow (A = B)$$

Clearly state whether this statement is true or false and include guiding text in your proof to indicate the proof strategies you're using.

3. (12 points) Give counterexamples that prove that each of the following statements is **false**. Include justification as to why your counterexamples prove this. Your justification will probably need to use the definition of the set operations in question.

- (a) For any sets A and B , $A \times B \subseteq (A \cup B)$.
- (b) For any sets A, B, C , if $A \cup C = B \cup C$ then $A = B$.
- (c) For any finite sets A and B , $|A \cup B| > |A|$.

4. (10 points) Find a formula for the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}$$

and prove that it is correct for all positive integers n .

Similar but different example, for reference: the sum $1 + 2 + 3 + \cdots + n$ is given by the formula $\frac{1}{2}n(n+1)$. To prove that this formula is correct, we proceed by mathematical induction over the set of positive integers. The basis step is when $n = 1$. Here, we want to show that $1 = \frac{1}{2}1(1+1)$. To prove this, we simplify the RHS $= \frac{1}{2} \cdot 2 = 1$, as required. For the induction step, let n be an arbitrary positive integer and (as the induction hypothesis), assume that $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$. We WTS that $1 + 2 + \cdots + (n+1) = \frac{1}{2}(n+1)((n+1)+1)$. By the induction hypothesis, we can rewrite the LHS as $1 + 2 + \cdots + (n+1) = 1 + 2 + \cdots + n + (n+1) = \frac{1}{2}n(n+1) + (n+1) = \frac{n(n+1)+2(n+1)}{2} = \frac{n^2+3n+2}{2}$. Factoring the numerator of this last expression, we see that $1 + 2 + \cdots + (n+1) = \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}$, as required. Thus the induction is complete and the formula is correct for all positive integers n .

5. (10 points) A guest at a party is a celebrity if this person is known by every other guest, but knows none of them. There is at most one celebrity at a party, for if there were two, they would know each other. A particular party may have no celebrity. Your assignment is to find the celebrity, if one exists, at a party, by asking only one type of question: asking a guest whether they know a second guest. Everyone must answer your questions truthfully. That is, if Alice and Bob are two people at the party, you can ask Alice whether she knows Bob; she must answer correctly. Use mathematical induction to show that if there are n people at the party (n is an integer greater than or equal to 2), then you can find the celebrity, if there is one, with (at most) $3(n-1)$ questions. [Hint: First ask a question to eliminate one person as a celebrity. Then use the inductive hypothesis to identify a potential celebrity. Finally, ask two more questions to determine whether that person is actually a celebrity.]