Tiling is not Decidable

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Previously, we showed that that there exist languages which are not decidable. However all such languages that we've seen are "metacomputing" languages, in the sense that they are languages defined by the descriptions of Turing machines with various properties. The goal of these lectures is to show that there exist other sorts of problems which are also undecidable.

Definition 1. A language L is in the class RE of recursive enumerable, or recognizable languages if there exists a Turing machine M such that for all $x \in L$, $M(x)$ halts and accepts, and for all $x \notin L$, $M(x)$ does not accept (it may reject, or it may not halt.)

Definition 2. A language L is in the class REC of recursive, or decidable languages if there exists a Turing machine M such that for all $x \in L$, $M(x)$ halts and accepts, and for all $x \notin L$, $M(x)$ halts and rejects.

Definition 3. A language L is in the class co-RE of co-recursive enumerable, or co-recognizable languages, if $\overline{L} = \{x \mid x \notin L\}$ is in RE.

Claim 1. $REC = RE \cap co\text{-}RE$

Definition 4. A *Turing reduction* from language A to a language B, denoted $A \leq_T B$, is an oracle Turing machine M such that M^B decides A.

Think of an oracle TM M^B as a regular TM which may use a decider for B as a subroutine as needed.

Definition 5. A *many-one reduction* or *map reduction* from a language A to a language B, denoted $A \leq_m B$, is a computable function f such that $x \in A \iff f(x) \in B$.

Let $TILING = \{\langle\tau\rangle \mid \tau$ is a set of simple rectilinear polygons such that there exists a square which may be exactly tiled using copies of polygons in τ without rotating them$\}$. See Figure 1 for an example.

We wish to show that $TILING$ is undecidable (not in REC). To do so, we will give a sequence of reductions from $HALT$, which we previously proved was undecidable: $HALT \leq_m U \leq_m U_\lambda \leq_m U'_\lambda \leq_m ABSTRACT-TILING \leq_m TILING$. Define the languages U, U_λ, and U'_λ as follows: $U = \{\langle M, w\rangle \mid M$

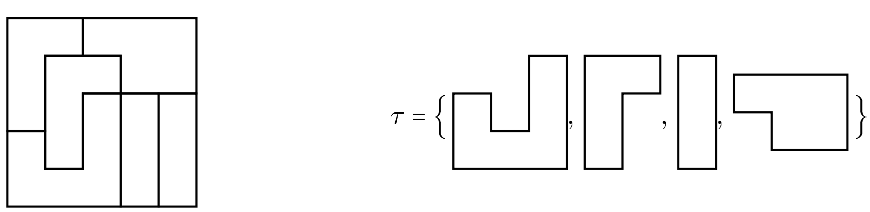

Figure 1: Tiling a square with tiles from τ.

is a TM and M halts and accepts $w\}$, $U_\lambda = \{\langle M\rangle \mid M$ is a TM which halts and accepts the empty string $\}$, and $U'_\lambda = \{\langle M\rangle \mid M$ is a TM which halts and accepts the empty string with a blank tape and the tape head at the beginning of the tape when it halts $\}$.

The first three reductions, $HALT \leq_m U \leq_m U_\lambda \leq_m U'_\lambda$, are straightforward and omitted. We will begin the actual proof by defining $ABSTRACT-TILING$ and showing $U'_\lambda \leq_m ABSTRACT-TILING$.

Let $ABSTRACT-TILING = \{\langle\Sigma,\tau\rangle \mid \Sigma$ is a set of symbols and τ is a set of 2×3 matrices with entries from $\Sigma' = \Sigma \cup \{\llcorner, \lrcorner, \ulcorner, \urcorner, -, \shortmid\}$ such that there exists a $T\times T$ matrix M, for some $T \geq 3$, with entries from Σ' satisfying (i) every 2×3 sub-matrix of M is in τ, (ii) the top row of M is of the form $\ulcorner, -, \ldots, -, \urcorner$, (iii) the bottom row of M is of the form $\llcorner, -, \ldots, -, \lrcorner$, (iv) the left edge of M is of the form $\ulcorner, \shortmid, \ldots, \shortmid, \llcorner$, and (v) the right edge of M is of the form $\urcorner, \shortmid, \ldots, \shortmid, \lrcorner$ $\}$.

Claim 2. $U'_\lambda \leq_m ABSTRACT-TILING$

Proof. For any TM M, we wish to construct a pair $\langle\Sigma,\tau\rangle$ such that $\langle M\rangle \in U'_\lambda \iff \langle\Sigma,\tau\rangle \in ABSTRACT-TILING$. Suppose that M halts and accepts the empty string (with a blank tape, and the tape head in the left position) in T times steps. We will think of a $T+2\times T+2$ matrix as encoding the computation history of M, with the edges of matrix consisting of the required boundary pieces. Each row of the matrix will correspond to the contents of the tape of M, where the first row corresponds to the initial tape, and the last row correspond to M in an accepting state with a blank tape. We will encode the position of the tape head and the state of M by using a pair of symbol and state for the current position of the tape head.

Let $\Sigma = \Sigma_M \cup \Sigma_M \times Q_M$, where Σ_M and Q_M are the tape alphabet and the set of states of M respectively.

$$\text{Let } \tau = \{\begin{pmatrix} \ulcorner & - & - \\ \shortmid & (\$, q_0) & \$ \end{pmatrix}, \begin{pmatrix} - & - & - \\ (\$, q_0) & \$ & \$ \end{pmatrix}, \begin{pmatrix} - & - & - \\ \$ & \$ & \$ \end{pmatrix}, \begin{pmatrix} - & - & \urcorner \\ \$ & \$ & \shortmid \end{pmatrix},$$

Corresponding to the 1st, 2nd, subsequent, and last sub-matrices in the first row, where q_0 is the start state of M and \$ is the blank symbol.

$$\begin{pmatrix} \shortmid & (\$, q_a) & \$ \\ \llcorner & - & - \end{pmatrix}, \begin{pmatrix} (\$, q_a) & \$ & \$ \\ - & - & - \end{pmatrix}, \begin{pmatrix} \$ & \$ & \$ \\ - & - & - \end{pmatrix}, \begin{pmatrix} \$ & \$ & \shortmid \\ - & - & \lrcorner \end{pmatrix},$$

Corresponding to the 1st, 2nd, subsequent, and last sub-matrices in the last row, where q_a is the accept state of M.

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & (\sigma_3, q) \end{pmatrix}, \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ (\sigma_1, q) & \sigma_2 & \sigma_3 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \sigma_2 & \shortmid \\ \sigma_1 & \sigma_2 & \shortmid \end{pmatrix}, \begin{pmatrix} \shortmid & \sigma_2 & \sigma_3 \\ \shortmid & \sigma_2 & \sigma_3 \end{pmatrix},$$

For all $\sigma_i \in \Sigma_M$ and $q \in Q_M$. These correspond to portions of the tape where the tape head doesn't start in any of the positions covered by the sub-matrix.

$$\begin{pmatrix} \sigma_1 & (\sigma_2, q) & \sigma_3 \\ \sigma_1 & (\sigma'_2, q') & \sigma_3 \end{pmatrix}, \begin{pmatrix} (\sigma_2, q) & \sigma_3 & \sigma_4 \\ (\sigma'_2, q') & \sigma_3 & \sigma_4 \end{pmatrix}, \begin{pmatrix} \sigma_0 & \sigma_1 & (\sigma_2, q) \\ \sigma_0 & \sigma_1 & (\sigma'_2, q') \end{pmatrix},$$

For all $\sigma_i \in \Sigma_M$ and $q \in Q_M$ such that M writes σ'_2, goes to state q' and doesn't move the tape head when in state q on symbol Γ_2.

$$\begin{pmatrix} \sigma_1 & (\sigma_2, q) & \sigma_3 \\ \sigma_1 & \sigma'_2 & (\sigma_3, q') \end{pmatrix}, \begin{pmatrix} (\sigma_2, q) & \sigma_3 & \sigma_4 \\ \sigma'_2 & (\sigma_3, q') & \sigma_4 \end{pmatrix}, \begin{pmatrix} \sigma_0 & \sigma_1 & (\sigma_2, q) \\ \sigma_0 & \sigma_1 & \sigma'_2 \end{pmatrix},$$

For all $\sigma_i \in \Sigma_M$ and $q \in Q_M$ such that M writes σ'_2, goes to state q' and moves the tape head right when in state q on symbol Γ_2.

In addition, we define a similar set of 2×3 sub-matrices for when the tape head moves left, and sub-matrices for left, right, and non-moving cases when the left edge or right edge is $|$.

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Next, we have to prove that $M \in U'_\lambda \iff \langle \Sigma, \tau \rangle \in ABSTRACT-TILING$. We prove this in 2 parts:

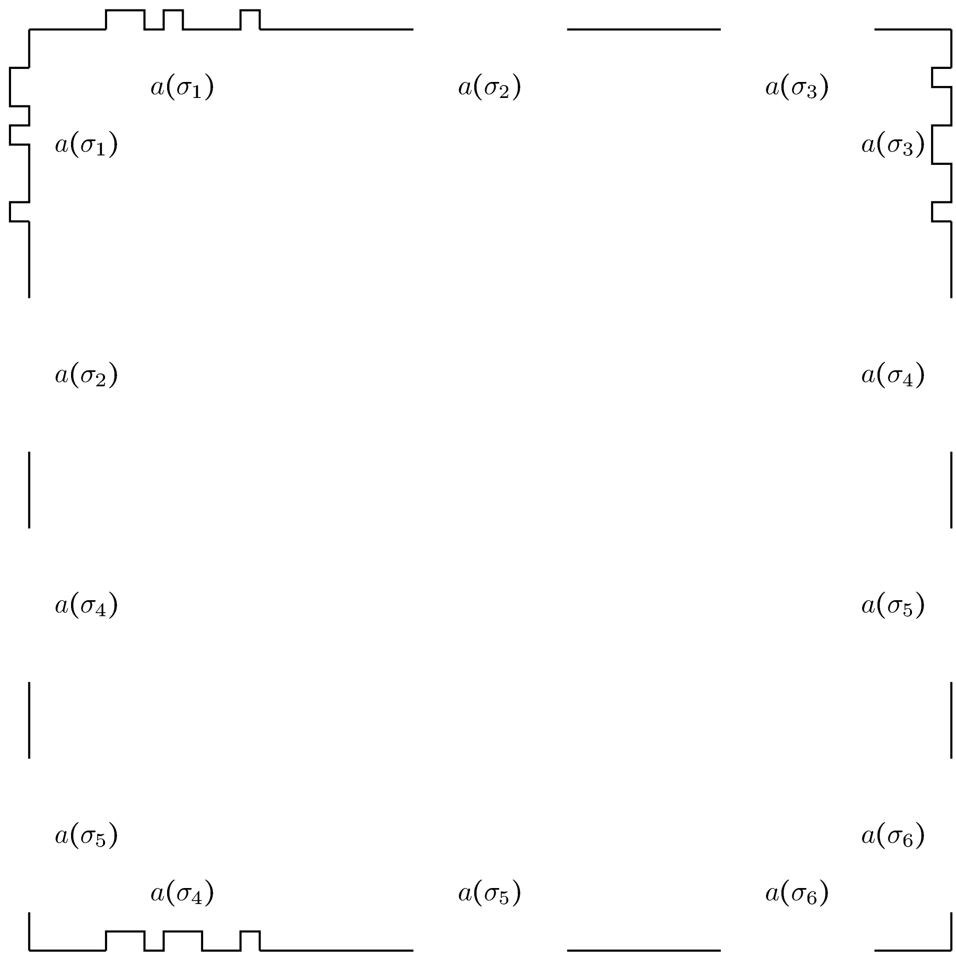
($\Longrightarrow$) Assume M halts and accepts with a blank tape, and the tape head in the left position in T steps. Let S be a $T+2 \times T+2$ matrix such that the edge pieces are as required, and $S[i+1, j+1]$ is the symbol in position j on the tape of M at time i, or the symbol and state if the tape head is in position j. By construction, such a matrix satisfies all of the conditions for $ABSTRACT-TILING$.

($\Longleftarrow$) Assume there exists a $T+2 \times T+2$ matrix S such that S satisfies all the conditions for $ABSTRACT-TILING$. For $i = 1$ and $i = T+2$, $S[i,j]$ must consist of edge pieces. For $i = 2$, the row $S[i,j]$ must be $|, (\$, q_0), \$, \ldots, \$, |$ by the first set of constraints. We argue by induction that for $i = 2, \ldots, T+1$, $S[i,j]$ must encode the tape in position $j-1$ and time $i-1$ since either the 3 positions around the tape head at the previous time step (row $i-1$ of S) are uniquely determined by the second row of the 2×3 sub-matrix of the form $\begin{pmatrix} \sigma_1 & (\sigma_2, q) & \sigma_3 \\ & \cdots & \end{pmatrix}$. All other positions are uniquely determined from the previous row by sub-matrices of the form $\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \sigma_2 & | \\ \sigma_1 & \sigma_2 & | \end{pmatrix}$, or $\begin{pmatrix} | & \sigma_2 & \sigma_3 \\ | & \sigma_2 & \sigma_3 \end{pmatrix}$. Finally, the last 2 rows of S must be $|, (\$, q_a), \$, \ldots, \$, |$ and $\llcorner, -, \ldots, -, \lrcorner$. Thus, S corresponds to the computation of M, and must end up in the accepting state with a blank tape. □

Finally, we prove $ABSTRACT-TILING \leq_m TILING$.

Let $a : \Sigma' \to \{0,1\}^k$ be a binary encoding of Σ' such that for all $\sigma \in \{\llcorner, \lrcorner, \ulcorner, \urcorner, -, |\}$, $a(\sigma) = 0^k$ and for all $\sigma_1 \neq \sigma_2 \in \Sigma$, $a(\sigma_1) \neq 0^k$, $a(\sigma_2) \neq 0^k$, and $a(\sigma_1) \neq a(\sigma_2)$.

For each $\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} \in \tau$, construct the following polygon:



Any time two horizontally polygon blocks are horizontally adjacent, the corresponding 2×3 rectangles must agree on the 4 symbols where they overlap, and vice versa when two sub-matrices overlap on 4 symbols. Similarly, when two polygon blocks are vertically adjacent, the corresponding 2×3 sub-matrices must agree on the overlapping 3 symbols.

This intuition can fairly straightforwardly be formalized into a rigorous proof.