CSE200 Lecture Notes – Tiling is not decidable

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1 Natural undecidable problems

Previously, we showed that that there exist languages which are not decidable. However all such languages that we've seen are "metacomputing" languages, in the sense that they are languages defined by the descriptions of Turing machines with various properties. The goal of these lectures is to show that there exist other sorts of problems which are also undecidable.

We show a "chain of reductions", starting from a "meta" problem, ending with a "natural combinatorial" problem.

$$Halt \leq_m SHalt \leq_m LSM \leq_m Tiling$$

Halt: meta problem
SHalt: constructed meta problem
LSM (Legal sub-matrix): uninteresting but combinatorial
Tiling: interesting and combinatorial

Previous lectures showed *Halt* is undecidable. In this lecture we introduce a reduction that "encodes" the computation of a Turing machine to geometric shapes, proving *Tiling* is undecidable.

The *Tiling* problem is to decide: Given some shapes, does there exist a square which may be exactly tiled using copies of these shapes without rotating them? Here we restrict the shapes to be *rectilinear polygons* (polygons with only vertical and horizontal edges).

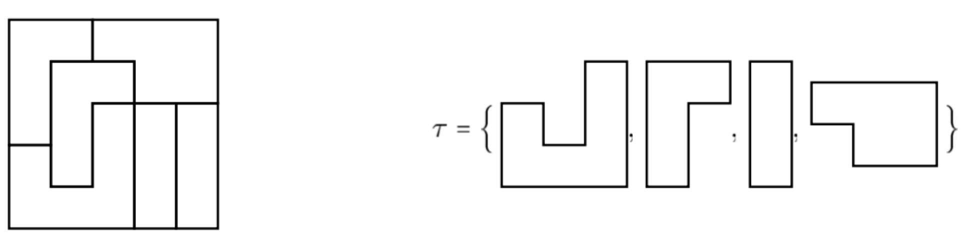


Figure 1: Tiling a square with tiles from τ.

The problem *SHalt* is a variation of the halting problem. $SHalt = \{M \mid M$ is a 1-tape TM with one halt state q_H, and M accepts the empty input λ with an empty tape.$\}$

2 Reduction from *Halt* to *SHalt*

To decide whether $(M, x) \in Halt$, we construct a M' so that $(M, x) \in Halt$ iff $M' \in SHalt$.

1. Construct $M_x =$
   - "Erase the input,
   - Write x,
   - Run M."
2. Convert M_x into a 1-tape TM M'_x.
3. Append a special halt state q_H and lines that say "Replace halt by erasing the tape and go to state q_H.

It is straightforward that

$$\begin{aligned}
&(M, x) \in Halt \\
\iff &M \text{ halts on } x \\
\iff &M_x \text{ halts on } \lambda \\
\iff &M'_x \text{ halts on } \lambda \\
\iff &M'_x \text{ halts on } \lambda \text{ with an empty tape on state } Q_H \\
\iff &M' \in SHalt
\end{aligned}$$

3 Reduction from *SHalt* to *LSM*

The *tableau* for M' is a matrix with T rows and T columns, where T is the running time of M'. At time i, location j, the symbol on the tableau is either the symbol on the j^{th}cell of tape at time i, if the head is not there, or $(q \to \sigma)$, where q is the state at time i and σ is the symbol on the j^{th}cell of tape at time i.

Intuitively, at time i, a symbol on the j^{th}tape cell is decided by the $j-1, j, j+1^{\text{th}}$symbols at time $i-1$. Because of this local property, we can create tiles of form

$\sigma_{i-1,j-1}$	$\sigma_{i-1,j}$	$\sigma_{i-1,j+1}$
	$\sigma_{i,j}$	

To represent the boundaries of the tableau, we add an extra row of blank symbols at the top and bottom of the tableau, and 2 columns of blank symbols on the left and right side.

Let Γ be the symbol set of M'. We extend the symbols to $\Gamma' = \Gamma \cup \{Q \to \Gamma\} \cup \{-, \$\}$, where "—" stands for the outer frame of the tableau, and "$" stands for an empty tape cell.

Definition of *LSM*

The *LSM* problem gives us

- Γ': set of symbols, including "—" (symbol for frame) and "\$" (symbol for empty tape cell).
- $\mathscr{L}$ of 2×3 matrices with entries from Γ'.

Question: Does there exist a $(T+2) \times (T+4)$ matrix τ with entries from Γ', so that its first and last rows, first two and last two columns are all "—", and every 2×3 consecutive sub matrix of $\tau \in \mathscr{L}$?

Construction of sub-matrices

First, we create sub-matrices for the top row of the tableau:

$$\begin{pmatrix} - & - & - \\ - & - & (q_0 \to \$) \end{pmatrix}, \begin{pmatrix} - & - & - \\ - & (q_0 \to \$) & \$ \end{pmatrix}, \begin{pmatrix} - & - & - \\ (q_0 \to \$) & \$ & \$ \end{pmatrix}, \begin{pmatrix} - & - & - \\ \$ & \$ & \$ \end{pmatrix}$$

$$\begin{pmatrix} - & - & - \\ \$ & \$ & - \end{pmatrix}, \begin{pmatrix} - & - & - \\ \$ & - & - \end{pmatrix}$$

Next, we create sub-matrices for the bottom row of the tableau.

$$\begin{pmatrix} - & - & (q_H \to \$) \\ - & - & - \end{pmatrix}, \begin{pmatrix} - & (q_H \to \$) & \$ \\ - & - & - \end{pmatrix}, \begin{pmatrix} (q_H \to \$) & \$ & \$ \\ - & - & - \end{pmatrix}, \begin{pmatrix} \$ & \$ & \$ \\ - & - & - \end{pmatrix}$$

$$\begin{pmatrix} \$ & \$ & - \\ - & - & - \end{pmatrix}, \begin{pmatrix} \$ & - & - \\ - & - & - \end{pmatrix}$$

Then, we create sub-matrices according to the rules of the TM.

No change at the current location:

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$$

Tape head enters from the left (or from the right):

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ (q \to \sigma_1) & \sigma_2 & \sigma_3 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & (q \to \sigma_3) \end{pmatrix}$$

Tape head is moving to the left:

$$\begin{pmatrix} \sigma_1 & \sigma_2 & (q \to \sigma_3) \\ \sigma_1 & (q' \to \sigma_2) & \sigma_3' \end{pmatrix}, \begin{pmatrix} \sigma_1 & (q \to \sigma_2) & \sigma_3 \\ (q' \to \sigma_1) & \sigma_2' & \sigma_3 \end{pmatrix}, \begin{pmatrix} (q \to \sigma_1) & \sigma_2 & \sigma_3 \\ \sigma_1' & \sigma_2' & \sigma_3 \end{pmatrix}$$

Tape head moving to the right is constructed symmetrically.

Proof of correctness

If τ is any matrix with the blank frame, and all 2×3 sub-matrices are in $\mathscr{L}$, then $\mathscr{L}$ is the tableau for M' (inside a frame). Define $\tau[i+1, j+2] = i^{\text{th}}$time, j^{th}place of tableau for M'.

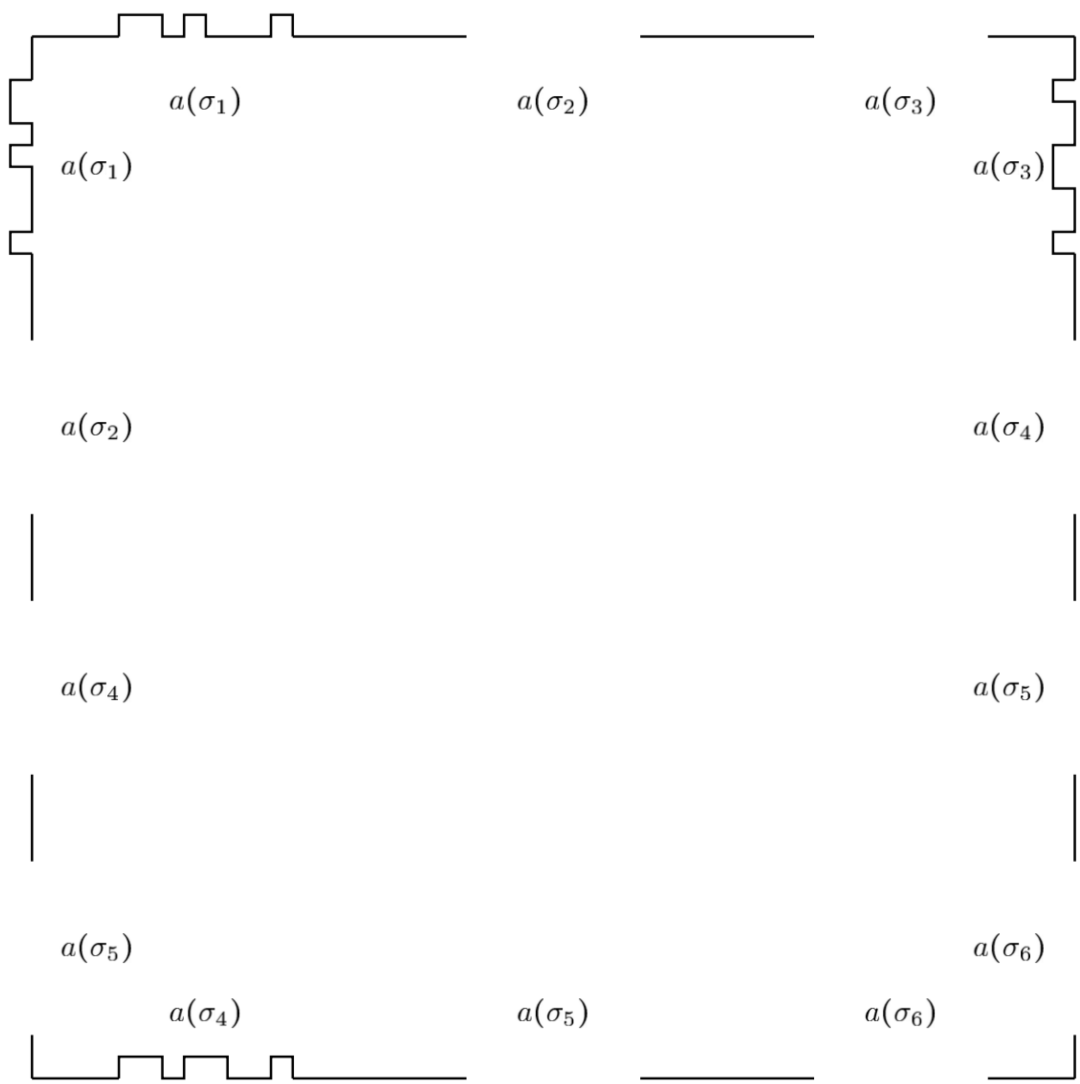
By induction on i, if j is within one of the position of tape head at time i, let j' be the position of the tape head at time i. The only possibilities is to have correct entries of tableau for $i+1$.

The second row is the first row of the tableau. If the i^{th}row is the i^{th}row of the tabuleau, we look at where the tape head is at the i^{th}step. From the construction above, the next row will have the corrrect symbols.

M' halts on λ iff there exists a matrix with all 2×3 sub-matrices in $\mathscr{L}$.

4 Reduction from *LSM* to *Tiling*

Let $a : \Gamma' \to \{0,1\}^k$ be a binary encoding of Γ' such that $a(-) = 0^k$, and for symbols σ other than "–", $a(\sigma) \neq 0^k$. For each matrix $\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_4 & \sigma_5 & \sigma_6 \end{pmatrix}$, construct the following polygon:



On the top and left sides, we create positive indents, while on the bottom and right sides, we make negative indents.

In the top left corner, the polygon must correspond to $\begin{pmatrix} - & - & - \\ - & - & \sigma_6 \end{pmatrix}$. Any time two polygons are horizontally adjacent, the corresponding 2×3 rectangles must agree on the 4 symbols where they overlap, and vice versa when to sub-matrices overlap on 4 symbols. Similarly, when to polygons are vertically adjacent, the corresponding sub-matrices must agree on the overlapping 3 symbols.

We need to show that there is a matrix that meets the definition of LSM if and only if there is a way to assemble these geometric tiles into a rectangle.

Say we have a matrix τ satisfying LSM conditions.

Then we define $T_{i,j}$ = Tile corresponding to

$$\begin{matrix} \tau[i,j] & \tau[i,j+1] & \tau[i,j+2] \\ \tau[i+1,j] & \tau[i+1,j+1] & \tau[i+1,j+2] \end{matrix}$$

. Because they all come from τ, these fit together.

$$\begin{pmatrix} \tau[i,j] & \sigma_2 & \sigma_3 \\ \sigma_4 & \sigma_5 & \sigma_6 \end{pmatrix} \quad \begin{pmatrix} \tau[i,j+1] = \sigma_2 & \sigma_3 & ? \\ \sigma_5 & \sigma_6 & ? \end{pmatrix} \quad \begin{pmatrix} \tau[i,j+2] = \sigma_3 & ? & ? \\ \sigma_6 & ? & ? \end{pmatrix}$$

$$\begin{pmatrix} \tau[i+1,j] = \sigma_4 & \sigma_5 & \sigma_6 \\ ? & ? & ? \end{pmatrix} \quad \begin{pmatrix} \tau[i+1,j+1] = \sigma_5 & \sigma_6 & ? \\ ? & ? & ? \end{pmatrix} \quad \begin{pmatrix} \tau[i+1,j+2] = \sigma_6 & ? & ? \\ ? & ? & ? \end{pmatrix}$$

In the reverse direction, say we have any way of assembling these tiles into a rectangle. The main areas of the tiles (not counting prongs and indents) only fit together if the tiles are arranged in a grid pattern, so we can call the tiles in the tiling of the rectangle $T_{i,j}$ according to their position in this grid. By construction, each tile corresponds to a 2×3 matrix of symbols. We define $\tau[i,j]$ as the first row first column symbol of the matrix corresponding to the i,j^{th}tile.

Because the exterior of the rectangle is flat, first row, first two columns, last two columns and last row are all blank.

Because prongs have to match indents on neighboring tiles, if we look at any 2 by 3 area of tiles, the first row first columns of all six tiles have to match the six entries of the first row, first column tile in the region. So each 2×3 sub-matrix of τ corresponds to a single tile, and so must be from the set of legal submatrices.

Thus, τ is matrix satisfying the conditions of LSM.