CSE 140, Lecture 2
Combinational Logic

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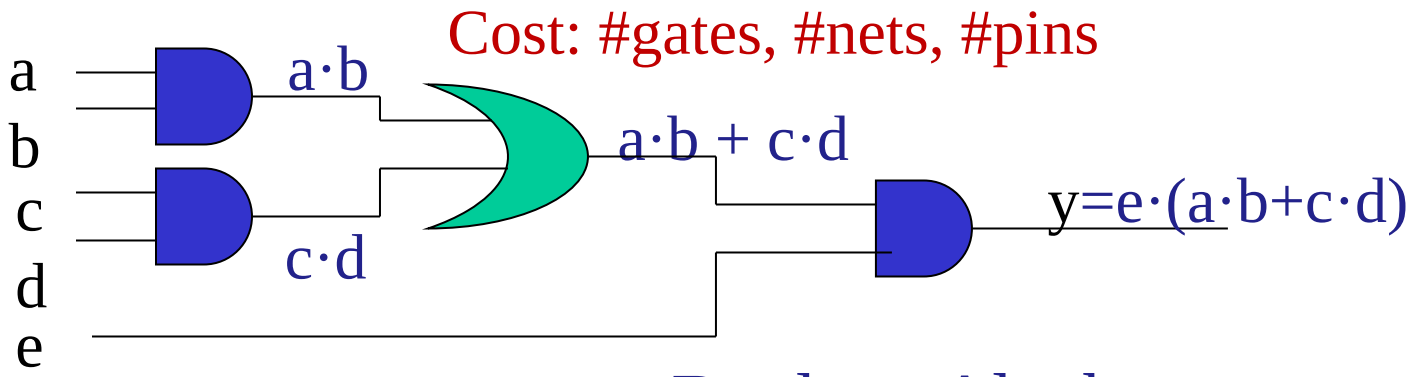
Combinational Logic Outlines

1. Introduction
    1. Scope
    2. Review of Boolean Algebra
    3. Review: Laws/Theorems and Digital Logic
2. Specification
3. Synthesis

1.1 Combinational Logic: Scope

- Description
  - Language: e.g. C Programming, Verilog, VHDL
  - Boolean algebra
  - Truth table
- Design
  - Schematic Diagram
  - Inputs, Gates, Nets, Outputs
- Goal
  - Validity: correctness, turnaround time
  - Performance: power, timing, cost
  - Testability: yield, diagnosis, robustness

Scope: Boolean Algebra for Logic



Schematic Diagram:

5 primary inputs

1 primary output

4 gates (3 ANDs, 1 OR)

9 signal nets

Boolean Algebra:

5 variables

1 expression

4 operators (3 ANDs, 1 OR)

5 literals

Scope: Boolean Algebra for Logic

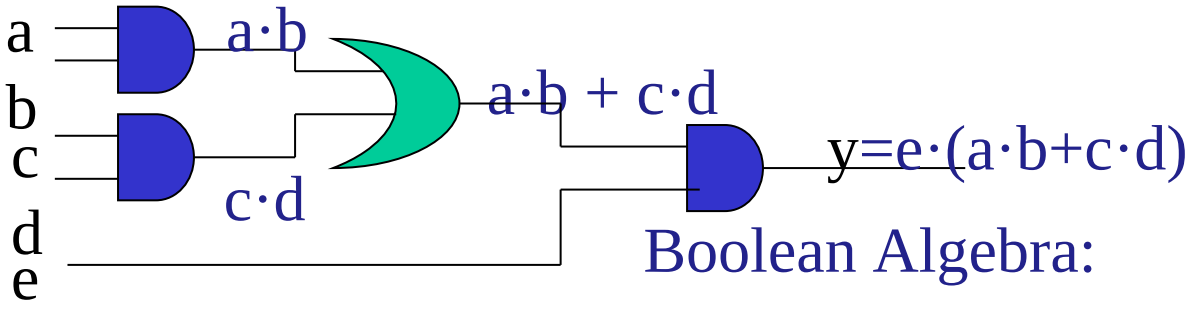
Schematic Diagram:

5 primary inputs

4 components (gates)

9 signal nets

12 pins



Boolean Algebra:

5 literals

4 operators

A. #inputs
B. #gates
C. #nets
D. #pins
E. None

I. #operators
II. #literals + #operators
III. #literals + 2 #operators - 1

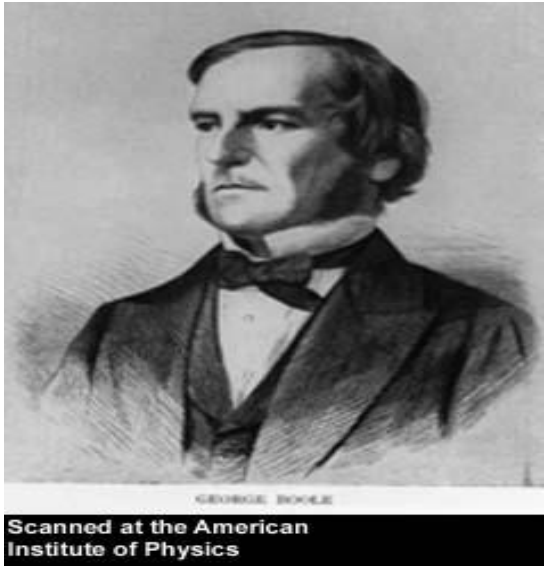
Schematic Diagram vs. Boolean Expression

- Boolean Expression: #literals, #operators
- Schematic Diagram: #gates, #nets, #pins

- One more example?

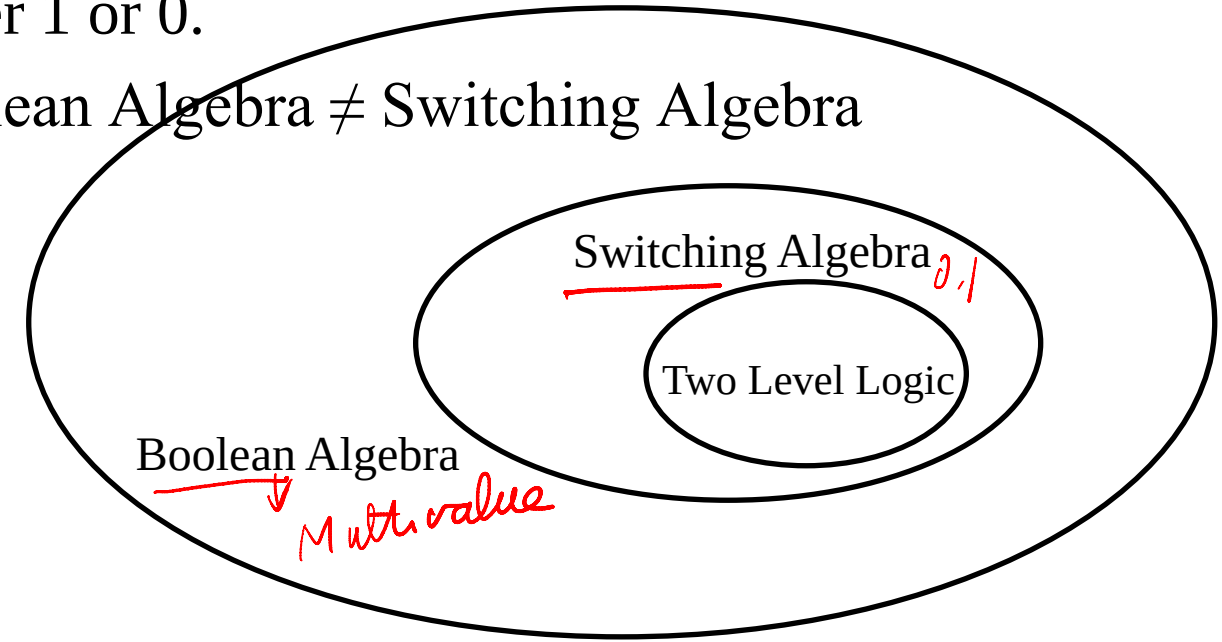
1.2 George Boole, 1815 - 1864

- Born to working class parents
- Taught himself mathematics and joined the faculty of Queen's College in Ireland.
- Wrote *An Investigation of the Laws of Thought* (1854)
- Introduced binary variables
- Introduced the three fundamental logic operations: AND, OR, and NOT



Switching Algebra

- Boolean Algebra: multiple-valued logic, i.e. each variable have multiple values.
- Switching Algebra: binary logic, i.e. each variable can be either 1 or 0.
- Boolean Algebra ≠ Switching Algebra



Scope: Binary Values

- Typically consider only two discrete values:
  - 1's and 0's
  - 1, TRUE, HIGH
  - 0, FALSE, LOW
- 1 and 0 can be represented by specific voltage levels, rotating gears, fluid levels, etc.
- Digital circuits usually depend on specific voltage levels to represent 1 and 0
- *Bit*: *B*inary dig*it*

Switching Algebra

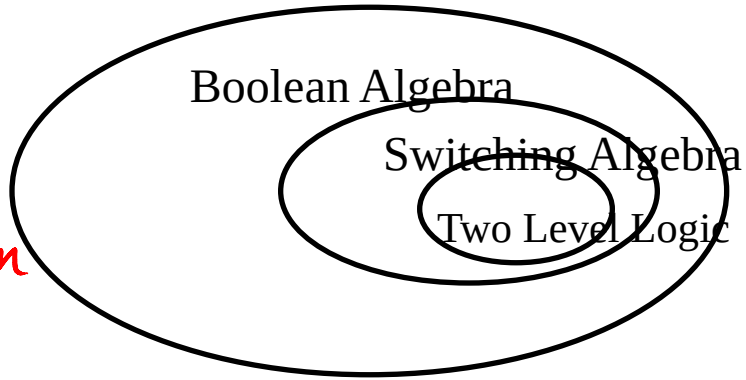
- Two Level Logic: Sum of products, or product of sums, e.g. ab + a'c + a'b', (a'+c)(a+b')(a+b+c')
- Multiple Level Logic: Many layers of two level logic with some inverters, e.g. (((a+bc)'+ab')+b'c+c'd)'bc+c'e

Features of Digital Logic Design

- Multiple Outputs
- Don't care sets

Handy Tools:

- DeMorgan's Law: Complements
- Consensus Theorem
- Shannon's Expansion
- Truth Table
- Karnaugh Map (single output, two level logic)



Review of Boolean Algebra

Let B be a nonempty set with two 2-input operations, a 1-input operation `, and two distinct elements 0 and 1. Then B is called a Boolean algebra if the following axioms hold.

- Commutative laws: $a+b=b+a$, $a\cdot b=b\cdot a$
- Distributive laws: $a+(b\cdot c)=(a+b)\cdot(a+c)$, $a\cdot(b+c)=a\cdot b+a\cdot c$
- Identity laws: $a+0=a$, $a\cdot 1=a$
- Complement laws: $a+a'=1$, $a\cdot a'=0$

Duality

(0,1) (AND OR)

$a \rightarrow a'$

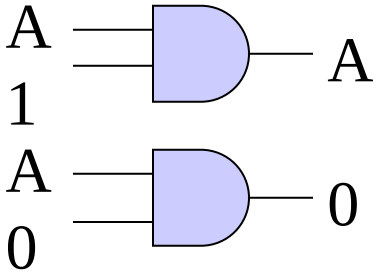
complement (one only one)

1.3 Boolean algebra and switching functions: Operators and Digital Logic Gates

Two-input operator
AND ($\wedge$, $\cdot$)

AND

A	B	Y
0	0	0
0	1	0
1	0	0
1	1	1

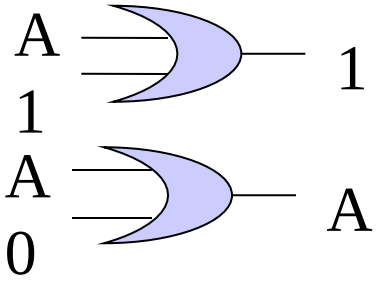


0 dominates in AND
0 blocks the output
1 passes signal A

Two-input operator
OR ($\vee$, +)

OR

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1



1 dominates in OR
1 blocks the output
0 passes signal A

One-input operator
NOT (Complement, `)

NOT

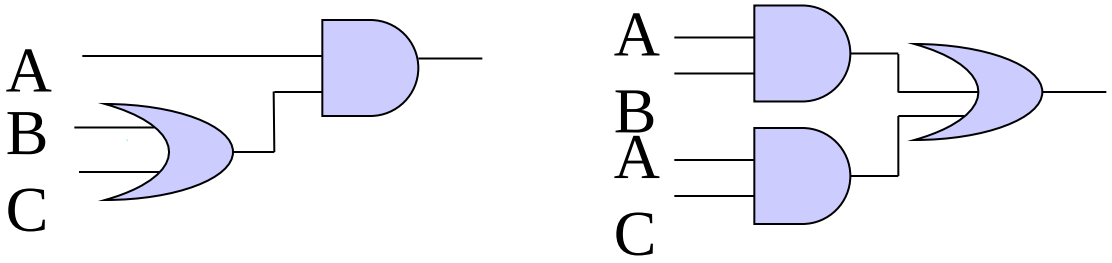
A	Y
0	1
1	0

Review: Laws and Digital Logic

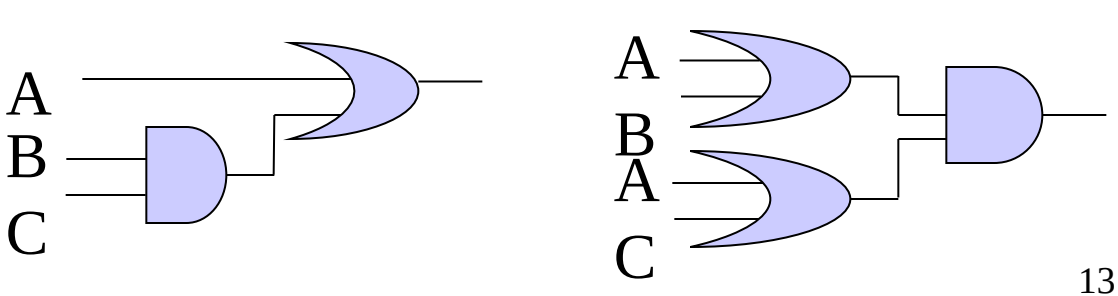
1. Identity $A * 1 = A$ $A + 1 = 1$
   $A * 0 = 0$ $A + 0 = A$
2. Complement $A + A' = 1$ $A * A' = 0$

T8. Distributive Law

$A(B+C) = AB + AC$



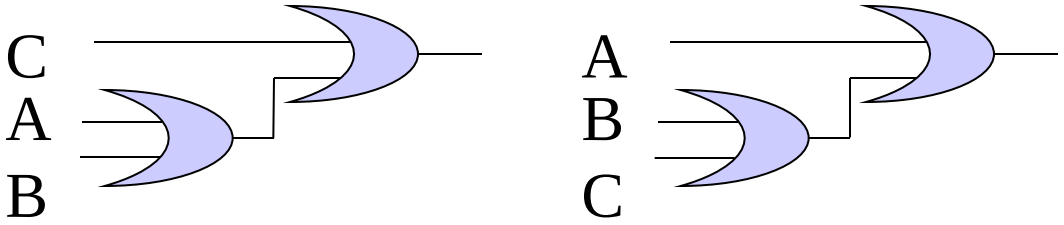
$A+BC = (A+B)(A+C)$



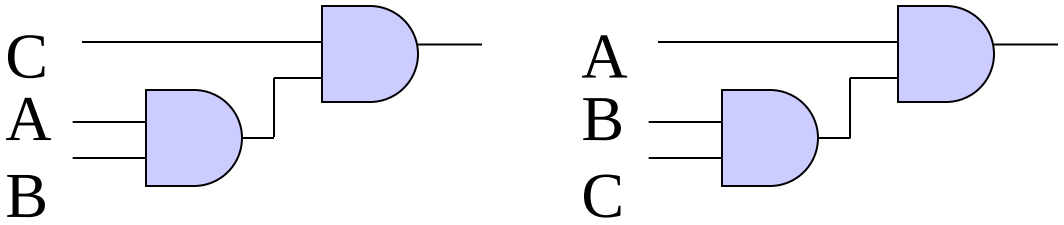
Review: Laws and Digital Logic

T7. Associativity

(A+B) + C = A + (B+C)



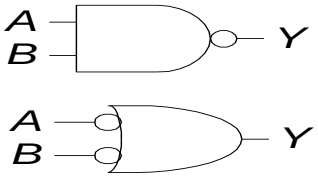
(AB)C = A(BC)



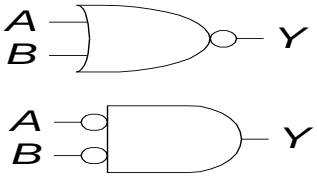
DeMorgan's Theorem and Digital Logic

T12. DeMorgan's Theorem (A+B)' = A'B' (AB)' = A' + B'

- $Y = (AB)' = A' + B'$

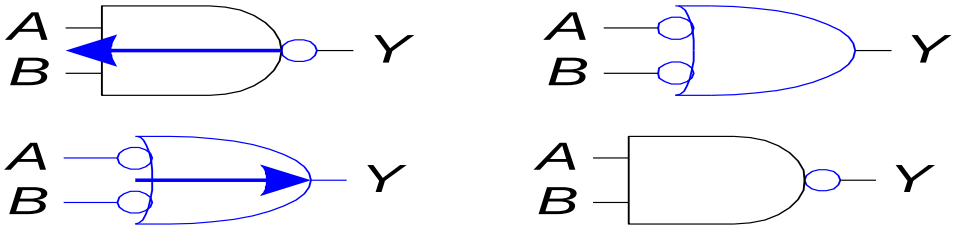


- $Y = (A + B)' = A'B'$



DeMorgan's Theorem: Bubble Pushing

- Pushing bubbles backward (from the output) or forward (from the inputs) changes the body of the gate from AND to OR or vice versa.
- Pushing a bubble from the output back to the inputs puts bubbles on all gate inputs.



- Pushing bubbles on *all* gate inputs forward toward the output puts a bubble on the output and changes the gate body.

Consensus Theorem

- AB+AC+B'C

  =AB+B'C

- (A+B)(A+C)(B'+C)

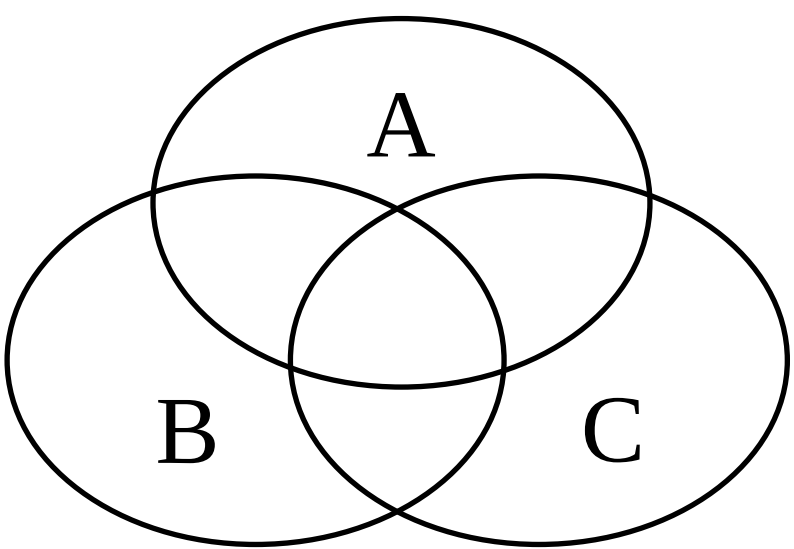
  =(A+B)(B'+C)

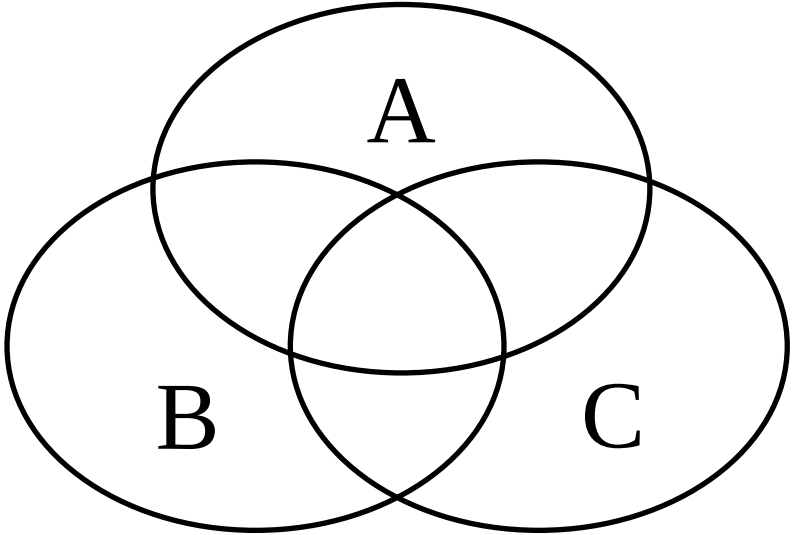
The consensus of AB, B'C is: ?

Exercise: to prove the reduction using
(1) Venn Diagrams,
(2) Boolean algebra,
(3) Logic simulation and
(4) Shannon's expansion

Consensus Theorem: Venn Diagrams

AB+AC+B’C : AB+B’C





Consensus Theorem: Boolean Algebra

- AB+AC+B'C

=AB+B'C

AB+AC+B'C

=AB+AC1+B'C

=AB+AC(B+B')+B'C

=AB+ABC+AB'C+B'C

=AB(1+C)+(A+1)B'C

=AB+B'C

- (A+B)(A+C)(B'+C)

=(A+B)(B'+C)

$f(a,b,c) = Bf(A,0,C) + B'f(A,1,C)$

~~$=(BA)+B'C$~~

$=AB+B'C$

Consensus Theorem: Logic Simulation

f(A,B,C)= AB+AC+B’C

g(A,B,C)=AB+B’C

Index	A	B	C	AB	AC	B’C	f	g
0	0	0	0	0	0	0		
1	0	0	1	0	0	1		
2	0	1	0	0	0	0		
3	0	1	1	0	0	0		
4	1	0	0	0	0	0		
5	1	0	1	0	1	1		
6	1	1	0	1	0	0		
7	1	1	1	1	1	0		

Shannon's Expansion

- Shannon's expansion assumes a switching algebra system
- Divide a switching function into smaller functions
- Pick a variable x, partition the switching function into two cases: x=1 and x=0
  - f(x,y,z,…)= xf(x=1,y,z,…) + x'f(x=0,y,z,…)
- For example
  - f(x)=xf(1)+x'f(0)
  - f(x,y)=xf(1,y)+x'f(0,y)

Shannon’s Expansion: Example

f(x,y,z)=xf(?,y,z)+x’f(?’,y,z)

A. ?=0

B. ?=1.

f(x,y)=(x+f(?,y))(x’+f(?’,y))

- A. ?=0
- B. ?=1.

Shannon's Expansion

- Decompose the switching function into minterms

$$f(x,y) = xf(1,y) + x'f(0,y)$$
$$= x\big(yf(1,1) + y'f(1,0)\big) + x'(y\big(f(0,1) + y'f(0,0)\big)$$
$$= xyf(1,1) + xy'f(1,0) + x'yf(0,1) + x'y'f(0,0).$$

- Decompose the switching function into maxterms

$$f(x,y) = (x' + f(1,y))(x + f(0,y))$$
$$= (x' + (y' + f(1,1)(y + f(1,0)))(x + (y' + f(0,1))(y + f(0,0)))$$
$$= (x' + y' + f(1,1))(x' + y + f(1,0))\big(x + y' + f(0,1)\big)\big(x + y + f(0,0)\big)$$

Shannon's Expansion: Example

Which term in AB'+AC+BC can be deleted?

A. AB'

B. AC

C. BC

D. None of the above

Review Summary: Switching Algebra and Karnaugh Map

Shannon's expansion and consensus theorem are used for logic optimization

- Shannon's expansion divides the problem into smaller functions
- Consensus theorem finds common terms when we merge small functions
- Karnaugh map mimics the above two operations in two dimensional space as a visual aid.

Part I. Combinational Logic

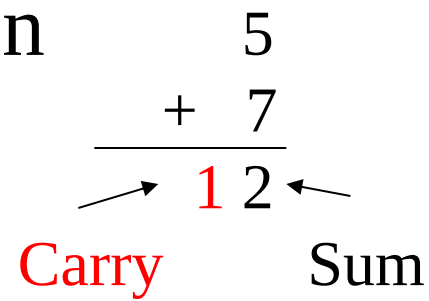
II) Specification

1. Language
2. Boolean Algebra

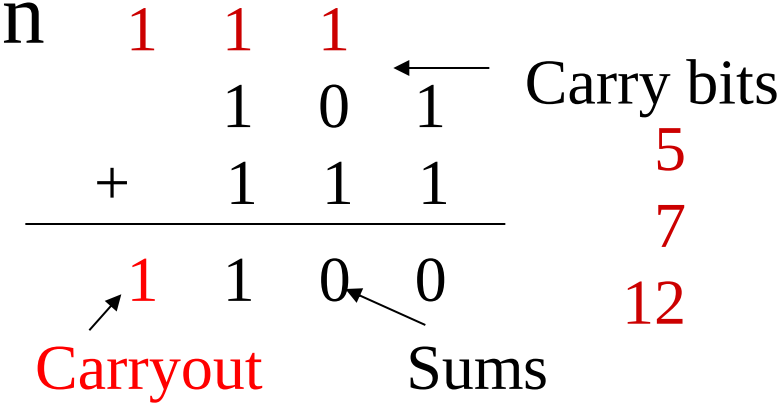
   Canonical Expression: Sum of minterms and Product of maxterms
3. Truth Table
4. Incompletely Specified Function

II. Specification

Decimal Addition



Binary Addition

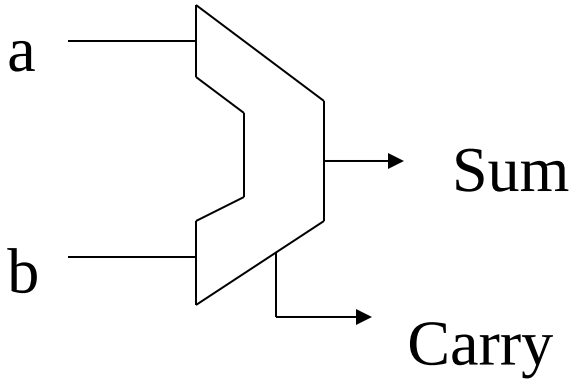


Binary Addition: Hardware

- Half Adder: Two inputs (a,b) and two outputs (carry, sum).

- Full Adder: Three inputs (a,b,c_{in}) and two outputs (carry, sum).

Half Adder



Truth Table

a	b	carry	sum
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Switching Function

Switching Expressions:

Sum (a,b) = a’·b + a·b’

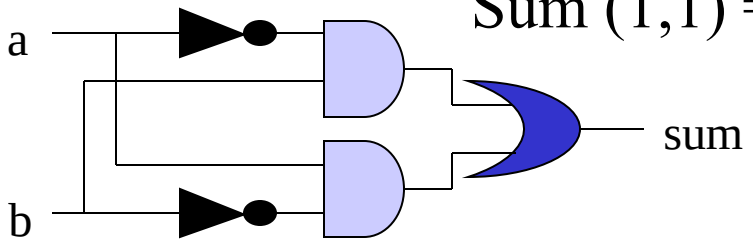
Carry (a, b) = a·b

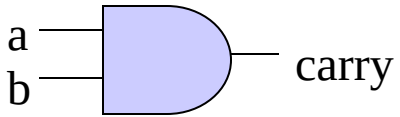
Ex:

Sum (0,0) = 0’·0 + 0·0’ = 0 + 0 = 0

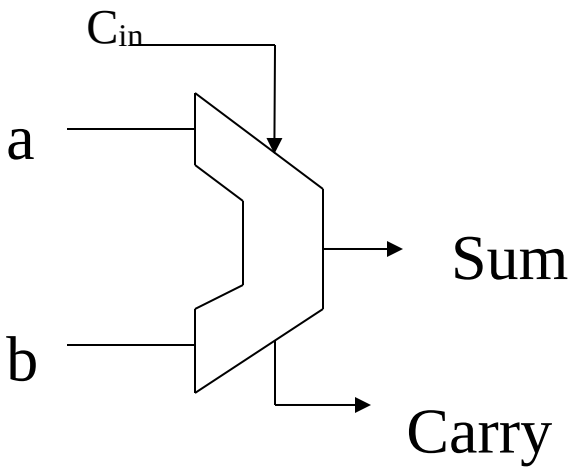
Sum (0,1) = 0’·1 + 0·1’ = 1 + 0 = 1

Sum (1,1) = 1’·1 + 1·1’ = 0 + 0 = 0





Full Adder



Truth Table

Id	a	b	c_{in}	carry	sum
0	0	0	0	0	0
1	0	0	1	0	1
2	0	1	0	0	1
3	0	1	1	1	0
4	1	0	0	0	1
5	1	0	1	1	0
6	1	1	0	1	0
7	1	1	1	1	1

Minterm and Maxterm

Id	a	b	c	carryout		
0	0	0	0	0		a+b+c
1	0	0	1	0		a+b+c’
2	0	1	0	0		a+b’+c
3	0	1	1	1	a’ b c	
4	1	0	0	0		a’+b+c
5	1	0	1	1	a b’c	
6	1	1	0	1	a b c’	
7	1	1	1	1	a b c	

maxterm

Minterms

$f_1(a,b,c) = a'bc + ab'c + abc' + abc$

$a'bc = 1$ iff $(a,b,c,) = (0,1,1)$
$ab'c = 1$ iff $(a,b,c,) = (1,0,1)$
$abc' = 1$ iff $(a,b,c,) = (1,1,0)$
$abc = 1$ iff $(a,b,c,) = (1,1,1)$
$f_1(a,b,c) = 1$ iff $(a,b,c) = (0,1,1), (1,0,1), (1,1,0),$ or $(1,1,1)$

Ex: $f_1(1,0,1) = 1'01 + 10'1 + 101' + 101 = 1$
$f_1(1,0,0) = 1'00 + 10'0 + 100' + 100 = 0$

Maxterms

$f_2(a,b,c) = (a+b+c)(a+b+c')(a+b'+c)(a'+b+c)$

a + b + c = 0 iff (a,b,c,) = (0,0,0)
a + b + c' = 0 iff (a,b,c,) = (0,0,1)
a + b' + c = 0 iff (a,b,c,) = (0,1,0)
a' + b + c = 0 iff (a,b,c,) = (1,0,0)
$f_2(a,b,c) = 0$ iff (a,b,c) = (0,0,0), (0,0,1), (0,1,0), (1,0,0)

Ex: $f_2(1,0,1) = (1+0+1)(1+0+1')(1+0'+1)(1'+0+1) = 1$
$f_2(0,1,0) = (0+1+0)(0+1+0')(0+1'+0)(0'+1+0) = 0$

$f_1(a,b,c) = a'bc + ab'c + abc' + abc$

$f_2(a,b,c) = (a+b+c)(a+b+c')(a+b'+c)(a'+b+c)$

$f_1(a, b, c) = m_3 + m_5 + m_6 + m_7 = \Sigma m(3,5,6,7)$

$f_2(a, b, c) = M_0M_1M_2M_4 = \Pi M(0, 1, 2, 4)$

iClicker: Does $f_1 = f_2$?

A. Yes

B. No.

The coverage of a single minterm. e.g. $m_4 = ab'c'$

Id	a	b	c_{in}	carry	minterm $_4 = ab'c'$
0	0	0	0	0	0
1	0	0	1	0	0
2	0	1	0	0	0
3	0	1	1	1	0
4	1	0	0	0	1
5	1	0	1	1	0
6	1	1	0	1	0
7	1	1	1	1	0

Only one row has a 1.

The coverage of a single maxterm. E.g. $M_4 = a'+b+c$

Id	a	b	c_{in}	carry	maxterm 4 = a+b+c
0	0	0	0	0	1
1	0	0	1	0	1
2	0	1	0	0	1
3	0	1	1	1	1
4	1	0	0	0	0
5	1	0	1	1	1
6	1	1	0	1	1
7	1	1	1	1	1

Only one row has a 0.

Incompletely Specified Function

Don’t care set is important because it allows us to minimize the function

Id	a	b	f (a, b)
0	0	0	1
1	0	1	0
2	1	0	1
3	1	1	-

1) The input does not happen.
2) The input happens, but the output is ignored.

Examples:

-Decimal number 0… 9 uses 4 bits. (1,1,1,1) does not happen.

-Final carry out bit (output is ignored).

Incompletely Specified Function

Id	a	b	c	g(a,b,c)
0	0	0	0	0
1	0	0	1	1
2	0	1	0	-
3	0	1	1	1
4	1	0	0	1
5	1	0	1	-
6	1	1	0	0
7	1	1	1	1

$g_1(a,b,c)=a'b'c+a'bc+ab'b'+abc$
$=m_1+m_3+m_4+m_7$
$=\sum m(1,3,4,7)$

$g_2(a,b,c)=(a+b+c)(a'+b'+c)$
$=M_0M_6$
$=\prod M(0,6)$

Exercise: Does $g_1(a,b,c) = g_2(a,b,c)$?