## Next

- More on recursion
- Higher-order functions
  - taking and returning functions
- Along the way, will see map and fold



## Tail Recursion: Factorial

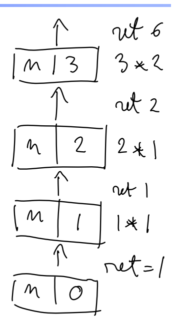
```
let rec fact n =
  if n<=0
  then 1
  else n * fact (n-1);;
```



## How does it execute?

```
let rec fact n =
  if n<=0
  then 1
  else n * fact (n-1);;

fac 3;;
```






## Tail recursion

- Tail recursion:
  - recursion where all recursive calls are immediately followed by a return
  - in other words: not allowed to do anything between recursive call and return



## Tail recursive factorial

```
let fact x =
```



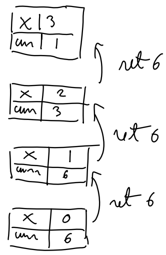
## Tail recursive factorial

```
let fact x =
   let rec helper x curr =
      if x <= 0
      then curr
      else helper (x - 1) (x * curr)
   in
      helper x 1;;
```

## How does it execute?

```
let fact x =
  let rec helper x curr =
    if x <= 0
    then curr
    else helper (x - 1) (x * curr)
  in
    helper x 1;;

fact 3;;
```



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## Tail recursion

- Tail recursion:
  - for each recursive call, the value of the recursive call is immediately returned
  - in other words: not allowed to do anything between recursive call and return
- Why do we care about tail recursion?
  - it turns out that tail recursion can be optimized into a simple loop

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## Compiler can optimize!

```
let fact x =
  let rec helper x curr =
    if x <= 0
    then curr
    else helper (x - 1) (x * curr)
  in
    helper x 1;;
```

recursion!

```
fact(x) {
  curr := 1;
  while (1) {
    if (x <= 0)
    then { return curr }
    else { x := x - 1; curr := (x * curr) }
}
```

Loop!

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## Tail recursion summary

- Tail recursive calls can be optimized as a jump
- Part of the language specification of some languages (ie: you can count on the compiler to optimize tail recursive calls)

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## max function

```
let max x y = if x < y then y else x;;

(* return max element of list l *)
let list_max l =
```

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## max function

```
let max x y = if x < y then y else x;;

(* return max element of list l *)
let list_max l =
  let rec helper curr l =
    match l with
      [] -> curr
    | h::t -> helper (max curr h) t
  in
    helper 0 l;;
```

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## concat function

```
(* concatenate all strings in a list *)
let concat l =
```

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## concat function

```
(* concatenate all strings in a list *)
let concat l =
   let rec helper curr l =
      match l with
        [] -> curr
      | h::t -> helper (curr ^ h) t
   in
      helper "" l;;
```

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## What's the pattern?

```
let list_max l =
   let rec helper curr l =
      match l with
         [] -> curr
       | h::t -> helper (max h curr) t
   in helper 0 l;;
```

```
let concat l =
   let rec helper curr l =
      match l with
        [] -> curr
      | h::t -> helper (curr ^ h) t
   in helper "" l;;
```

## fold, the general helper func!

```
(* to help us see the pattern: *)
let list_max l =
   let rec helper curr l =
      match l with
         [] -> curr
       | h::t -> helper (max h curr) t
   in helper 0 l;;
```

```
(* fold, the coolest function there is! *)
let rec fold f curr l =
```

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## fold

```
(* fold, the coolest function there is! *)
let rec fold f curr l =
   match l with
      [] -> curr
    | h::t -> fold f (f curr h) t;;
```

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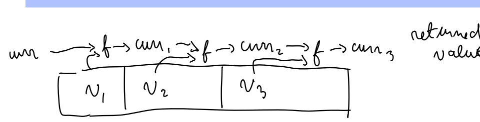
## fold

```
(* fold, the coolest function there is! *)
let rec fold f curr l =
   match l with
      [] -> curr
    | h::t -> fold f (f curr h) t;;
```

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## Examples of fold

```
let list_max =

let concat =

let multiplier =
```



## Examples of fold

```
let list_max = fold max 0;;

let concat = fold (^) "";;

let multiplier = fold (*) 1;;
```



## Examples of fold

```
let fact n =
  multiplier (interval 1 n);;
```

Notice how all the recursion is buried inside two functions: interval and fold!



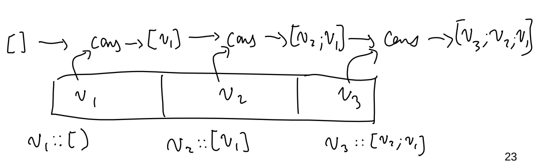
## Examples of fold

```
let cons x y = y::x;;
let f = fold cons [];;
(* same as:
   let f l = fold cons [] l *)
```



## Examples of fold

```
let cons x y = y::x;;
let f = fold cons [];;
(* same as:
   let f l = fold cons [] l *)
```






## More recursion: interval

```
(* return a list that contains
  the integers i through j
  inclusive *)
let rec interval i j =
```

## interval

```
(* return a list that contains
  the integers i through j
  inclusive *)
let rec interval i j =
  if  i > j
  then []
  else i::(interval (i+1) j);;
```

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## interval function with init fn

```
(* return a list that contains the
  elements f(i), f(i+1), ... f(j) *)
let rec interval_init i j f =
```

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## interval function with init fn

```
(* return a list that contains the
  elements f(i), f(i+1), ... f(j) *)
let rec interval_init i j f =
  if  i > j
  then []
  else (f i)::(interval_init (i+1) j f);;
```

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## interval function again

```
(* our regular interval function in terms
  of the one with the init function *)
let rec interval i j =
```

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## interval function again

```
(* our regular interval function in terms
  of the one with the init function *)
let rec interval i j =
  interval_init i j (fun x -> x);;
```

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## Interval function yet again!

```
(* let's change the order of parameters... *)
let rec interval_init f i j =
  if  i > j
  then []
  else (f i)::(interval_init f (i+1) j);;

(* now can use currying to get interval function! *)
let interval = interval_init (fun x -> x);;
```

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## Function Currying

In general, these two are equivalent:

```
let f = fun x1 -> … -> fun xn -> e
```

```
let f x1 … xn = e
```

Multiple argument functions by returning a function that takes the next argument

- Named after a person (Haskell Curry)



## Function Currying vs tuples

| | fn definition | fn call |
|---|---|---|
| Tuple version: | `let f (x1,…,xn) = e` | `f (x1,…,xn)` |
| Curried version: | `let f x1 … xn = e` | `f x1 … xn` |



## Function Currying vs tuples

Consider the following:

```
let lt x y = x < y;
```

Could have done: `let lt (x,y) = x<y;`

- But then no "testers" possible

In general: Currying allows you to **set** just the first n params (where n smaller than the total number of params)



## map

```
(* return the list containing f(e) for each
  element e of l *)
let rec map f l =
```



## map

```
(* return the list containing f(e) for each
  element e of l *)
let rec map f l =
   match l with
      [] -> []
   |  h::t -> (f h)::(map f t);;
```



## map

```
let incr x = x+1;;

let map_incr = map incr;;
map_incr (interval (-10) 10);;
```

## composing functions

(f ∘ g) (x) = f(g(x))

```
(* return a function that given an argument
  x applies f2 to x and then applies f1 to
  the result*)
let compose f1 f2 =
```

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## composing functions

(f ∘ g) (x) = f(g(x))

```
(* return a function that given an argument
  x applies f2 to x and then applies f1 to
  the result*)
let compose f1 f2 = fun x -> (f1 (f2 x));;

(* another way of writing it *)
let compose f1 f2 x = f1 (f2 x);;
```

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## Higher-order functions!

```
let map_incr_2 = compose map_incr map_incr;;
map_incr_2 (interval (-10) 10);;

let map_incr_3 = compose map_incr map_incr_2;;
map_incr_3 (interval (-10) 10);;

let map_incr_3_pos = compose pos_filer map_incr_3;;
map_incr_3_pos (interval (-10) 10);;
(compose map_incr_3 pos_filer) (interval (-10) 10);;
```

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## Higher-order functions!

```
let map_incr_2 = compose map_incr map_incr;;
map_incr_2 (interval (-10) 10);;

let map_incr_3 = compose map_incr map_incr_2;;
map_incr_3 (interval (-10) 10);;

let map_incr_3_pos = compose pos_filer map_incr_3;;
map_incr_3_pos (interval (-10) 10);;
(compose map_incr_3 pos_filer) (interval (-10) 10);;
```

Instead of manipulating lists, we are manipulating the list manipulators!

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## Exercise 1

```
let rec filter f l =
   match l with
     | [] -> []
     | h::t -> let t' = filter f t in
             if f h then h::t' else t'

let neg f x = not (f x)

let partition f l = (filter f l, filter (neg f) l)
```

This implementation is not ideal, since it unnecessarily processes the list twice. Rewrite partition so that it is a single call to fold_left, so the input list is processed only once. Recall:

```
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a
```

## Exercise 1

```
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a

let partition f l =
```

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## Exercise 1 Solution

```
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a

let partition f l =
   let fold_fn (pass,passnot) x =
        if f x then (pass@[x], passnot)
               else (pass, passnot@[x])
   in
   List.fold_left fold_fn ([],[]) l;;
```

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## Exercise 2

```
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a
val map : ('a -> 'b) -> 'a list -> 'b list
```

Implement map using fold:

```
let map f l =
```

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## Exercise 2 Solution

```
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a
val map : ('a -> 'b) -> 'a list -> 'b list
```

Implement map using fold:

```
let map f l =
   List.fold_left (fun acc x -> acc@[f x]) [] l
```

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## Benefits of higher-order functions

Identify common computation "patterns"

- Iterate a function over a set, list, tree …
- Accumulate some value over a collection

Pull out (factor) "common" code:

- Computation Patterns
- Re-use in many different situations

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## Funcs taking/returning funcs

Higher-order funcs enable modular code

- Each part only needs local information

Data Structure Client Uses `list` ⟷ Data Structure Library `list`

Uses meta-functions: `map,fold,filter`
With locally-dependent funs `(lt h)`, `square` etc.
Without requiring Implement. details of data structure

Provides meta-functions: `map,fold,filter`
to traverse, accumulate over lists, trees etc.
Meta-functions don't need client info

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## Different way of thinking

*"Free your mind"*
-Morpheus

- Different way of thinking about computation
- Manipulate the manipulators

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