

Lecture 6: Three and Four Input K-Maps

CSE 140: Components and Design Techniques for
Digital Systems

Diba Mirza

Dept. of Computer Science and Engineering

University of California, San Diego

What you should know at the end of this lecture....

1. How to simplify combinational circuits in three and four variables using K-maps
 - Definitions: Implicants, Prime Implicants, Essential prime implicants (also identifying these on the map)
 - A systematic approach to reduction using K-maps
2. Boolean Algebra vs K-maps
 - Algebra: Variables, product terms, minterms
 - K-map: Cells, adjacency

Truth table

Id	a	b	c	f(a,b,c)
0	0	0	0	1
1	0	0	1	0
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	0
6	1	1	0	1
7	1	1	1	0

Three variable K-maps

		(a,b)			
		(0,0)	(0,1)	(1,0)	(1,1)
c	0	0 1	2 1	4 1	6 1
	1	1 0	3 0	5 0	7 0

Is the above K-map correct?

- A. Yes
- B. No

Truth table

Id	a	b	c	f(a,b,c)
0	0	0	0	1
1	0	0	1	0
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	0
6	1	1	0	1
7	1	1	1	0

Reduction in three variable K-maps

Example 1

		(a,b)			
		(0,0)	(0,1)	(1,1)	(1,0)
c	= 0	0 1	2 1	6 1	4 1
	= 1	1 0	3 0	7 0	5 0

Truth table

Id	a	b	c	f(a,b,c)
0	0	0	0	0
1	0	0	1	0
2	0	1	0	1
3	0	1	1	1
4	1	0	0	0
5	1	0	1	0
6	1	1	0	1
7	1	1	1	1

Reduction in three variable K-maps

Example 2

		(a,b)			
		(0,0)	(0,1)	(1,1)	(1,0)
c	= 0	0 0	2 1	6 1	4 0
	= 1	1 0	3 1	7 1	5 0

Truth table

Id	a	b	c	f(a,b,c)
0	0	0	0	1
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	0
7	1	1	1	0

Three variable K-maps

Example 3

(a,b)		(0,0)	(0,1)	(1,1)	(1,0)
c	0	⁰ 1	² 0	⁶ 0	⁴ 1
	1	¹ 1	³ 0	⁷ 0	⁵ 1

Another 3-Input truth table

Id	a	b	c	f(a,b,c)
0	0	0	0	0
1	0	0	1	0
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	X
7	1	1	1	1

Corresponding K-map

	(0,0)	(0,1)	(1,1)	(1,0)
c = 0	⁰ 0	² 1	⁶ X	⁴ 1
c = 1	¹ 0	³ 0	⁷ 1	⁵ 1

Another 3-Input truth table

Id	a	b	c	$f(a,b,c)$
0	0	0	0	0
1	0	0	1	0
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	X
7	1	1	1	1

Corresponding K-map

		$b = 1$			
		(0,0)	(0,1)	(1,1)	(1,0)
$c = 0$		0 0	2 1	6 X	4 1
$c = 1$		1 0	3 0	7 1	5 1
		$a = 1$			

$$f(a,b,c) = a + bc'$$

Proof of Consensus Theorem using K Maps

Consensus Theorem: $A'B + AC + BC = A'B + AC$

<i>A</i>	<i>B</i>	<i>C</i>	<i>Y</i>
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

<i>Y</i> <i>C</i>	<i>AB</i>			
	00	01	11	10
0	0	1	0	0
1	0	1	1	1

<i>Y</i> <i>C</i>	<i>AB</i>			
	00	01	11	10
0	$\bar{A}\bar{B}\bar{C}$	$\bar{A}B\bar{C}$	$AB\bar{C}$	$A\bar{B}\bar{C}$
1	$\bar{A}\bar{B}C$	$\bar{A}BC$	ABC	$A\bar{B}C$

4-input K-map

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>Y</i>
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

<i>Y</i>		<i>AB</i>			
<i>CD</i>		00	01	11	10
	00				
	01				
	11				
	10				

4-input K-map

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>Y</i>
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

<i>Y</i> <i>CD</i> \ <i>AB</i>		00	01	11	10
		00	01	11	10
00		1	0	0	1
01		0	1	0	1
11		1	1	0	0
10		1	1	0	1

4-input K-map

- Identify adjacent cells containing 1's
- What happens when we combine these cells?

Y $CD \backslash AB$					
		00	01	11	10
CD	00	1	0	0	1
	01	0	1	0	1
	11	1	1	0	0
	10	1	1	0	1

4-input K-map

- Suppose $f(A'B', C'D)=1$, can we group 6 cells together?

$\begin{matrix} Y \\ \swarrow \end{matrix}$ $CD \backslash AB$		AB			
		00	01	11	10
00	00	1	0	0	1
01	01	0	1	0	1
11	11	1	1	0	0
10	10	1	1	0	1

Boolean Expression	K-Map
Variable x_i and its compliment x_i'	$\Leftrightarrow$ Two half planes Rx_i , and Rx_i'
Product term P (Px_i^* e.g. $b' c'$)	$\Leftrightarrow$ Intersect of Rx_i^* for all i in P e.g. Rb' intersect Rc'
Each minterm	$\Leftrightarrow$ One element cell
Two minterms are adjacent iff they differ by one and only one variable, eg: $abc' d, abc' d'$	$\Leftrightarrow$ The two cells are neighbors
Each minterm has n adjacent minterms	$\Leftrightarrow$ Each cell has n neighbors

Procedure for finding the minimal function via K-maps (layman terms)

1. Convert truth table to K-map
2. Group adjacent ones: In doing so include the largest number of adjacent ones (Prime Implicants)
3. Create new groups to cover all ones in the map: create a new group only to include at least once cell (of value 1) that is not covered by any other group
4. Select the groups that result in the minimal sum of products (we will formalize this because its not straightforward)

		Y			
		AB			
CD		00	01	11	10
	00	1	0	0	1
	01	0	1	0	1
	11	1	1	0	0
	10	1	1	0	1

Definitions: implicant, prime implicant, essential prime implicant

- Implicant: A product term that has non-empty intersection with on-set F and does not intersect with off-set R .
- Prime Implicant: An implicant that is not a proper subset of any other implicant i.e. it is not completely covered by any single implicant
- Essential Prime Implicant: A prime implicant with atleast one element that is not covered by one or more prime implicants.

Definition: Implicant

- Implicant: A product term that has non-empty intersection with on-set F and does not intersect with off-set R

Y CD \ AB	00	01	11	10
00	1	0	0	1
01	0	1	0	1
11	1	1	0	0
10	1	1	0	1

Q: Is this an implicant?

- A. Yes
B. No

Definition: Implicant

- Implicant: A product term that has non-empty intersection with on-set F and does not intersect with off-set R

		AB			
		00	01	11	10
CD	00	1	0	0	1
	01	0	1	0	1
	11	1	1	0	0
	10	1	1	0	1

Q: Is this an implicant?

- A. Yes
B. No

Definition: Prime Implicant

1. Implicant: Any product term that has non empty intersection with onset F and does not intersect with offset R or any group of adjacent ones in a K-map
2. Prime Implicant: A group of adjacent ones that cannot be fully covered by any other large group of ones

Y CD \ AB				
	00	01	11	10
00	1	0	0	1
01	0	1	0	1
11	1	1	0	0
10	1	1	0	1

Q: Is this a prime implicant?

- A. Yes
B. No

Definition: Prime Implicant

- Implicant: A product term that has non-empty intersection with on-set F and does not intersect with off-set R
- Prime Implicant: An implicant that is not a proper subset of any other implicant i.e. it is not completely covered by any single implicant

Y CD \ AB	00	01	11	10
00	1	0	0	1
01	0	1	0	1
11	1	1	0	0
10	1	1	0	1

Q: Is this a prime implicant?

- A. Yes
B. No

Definition: Essential Prime

1. Prime Implicant: A group of adjacent ones that cannot be fully covered by any other large group of ones

		Y			
		AB			
CD		00	01	11	10
	00	1	0	0	1
	01	0	1	0	1
	11	1	1	0	0
	10	1	1	0	1

Q: Are the corner four prime implicants?

- A. Yes
B. No

Definition: Essential Prime

1. Prime Implicant: A group of adjacent ones that cannot be fully covered by any other large group of ones
2. Essential Prime Implicant or Essential Prime: Prime implicants covering elements in the onset that contain at least one element that cannot be covered by any other prime implicant

Karnaugh map for function Y:

Y	AB			
CD \ AB	00	01	11	10
00	1	0	0	1
01	0	1	0	1
11	1	1	0	0
10	1	1	0	1

Definition: Essential Prime

1. Prime Implicant: A group of adjacent ones that cannot be fully covered by any other large group of ones
2. Essential Prime Implicant or Essential Prime: Prime implicants covering elements in the onset that contain at least one element that cannot be covered by any other prime implicant

Y CD \ AB		AB			
		00	01	11	10
CD	00	1	0	0	1
	01	0	1	0	1
	11	1	1	0	0
	10	1	1	0	1

Q: Are the corner four essential prime implicants?

- A. Yes
B. No

Include all essential prime implicants!

Y

CD \ AB	00	01	11	10
00	1	0	0	1
01	0	1	0	1
11	1	1	0	0
10	1	1	0	1

$$Y = \bar{A}\bar{C} + \bar{A}BD + A\bar{B}\bar{C} + \bar{B}\bar{D}$$

Reading the reduced K-map

Y $CD \backslash AB$					
		00	01	11	10
00	1	0	0	1	
01	0	1	0	1	
11	1	1	0	0	
10	1	1	0	1	

$$Y = \bar{A}\bar{C} + \bar{A}BD + A\bar{B}\bar{C} + \bar{B}\bar{D}$$

Definition: Non-Essential Prime

Non Essential Prime Implicant : Prime implicant that has no element that cannot be covered by other prime implicant

Q: Is the blue group a non-essential prime implicant?

ab \ cd	00	01	11	10
00	1		1	1
01		1	1	
11			1	1
10	1		1	1

- A. Yes
- B. No

Definition: Non-Essential Prime

Non Essential Prime Implicant : Prime implicant that has no element that cannot be covered by other prime implicant

Q: Is the blue group a non-essential prime implicant?

ab \ cd	00	01	11	10
00	1		1	1
01		1	1	
11			1	1
10	1		1	1

- A. Yes
- B. No

Procedure for finding the minimal function via K-maps (formal terms)

1. Convert truth table to K-map
2. Include all essential primes
3. Include non essential primes as needed to completely cover the on-set (all cells of value one)

$\begin{matrix} Y \\ \swarrow \\ CD \end{matrix}$		AB			
		00	01	11	10
00	1	0	0	1	
01	0	1	0	1	
11	1	1	0	0	
10	1	1	0	1	

K-maps with Don't Cares

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>Y</i>
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	X
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

<i>Y</i>		<i>AB</i>			
<i>CD</i>		00	01	11	10
	00				
01					
11					
10					

K-maps with Don't Cares

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>Y</i>
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	X
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

<i>Y</i> <i>CD</i> \ <i>AB</i>		00	01	11	10
		00	01	11	10
00	00	1	0	X	1
01	01	0	X	X	1
11	11	1	1	X	X
10	10	1	1	X	X

K-maps with Don't Cares

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>Y</i>
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	X
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

<i>Y</i>		<i>AB</i>			
<i>CD</i>		00	01	11	10
		00	01	11	10
00		1	0	X	1
01		0	X	X	1
11		1	1	X	X
10		1	1	X	X

$$Y = A + \bar{B}\bar{D} + C$$

Reducing Canonical expressions

Given $F(a,b,c,d) = \sum m(0, 1, 2, 8, 14)$

$D(a,b,c,d) = \sum m(9, 10)$

1. Draw K-map

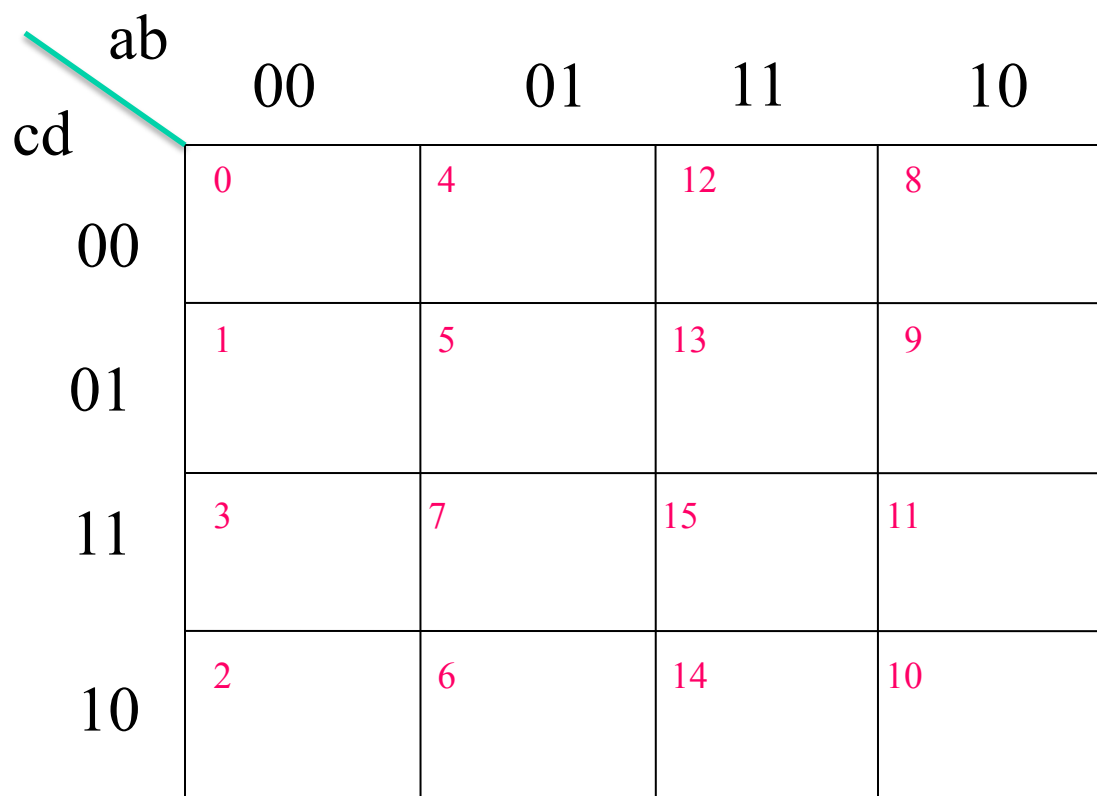
	ab	00	01	11	10
cd	00				
	01				
	11				
	10				

Reducing Canonical Expressions

Given $F(a,b,c,d) = \sum m(0, 1, 2, 8, 14)$

$D(a,b,c,d) = \sum m(9, 10)$

1. Draw K-map



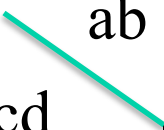
ab \ cd	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

Reducing Canonical Expressions

Given $F(a,b,c,d) = \sum m(0, 1, 2, 8, 14)$

$\mathcal{D}(a,b,c,d) = \sum m(9, 10)$

1. Draw K-map



ab		00	01	11	10
cd	00	⁰ 1	⁴ 0	¹² 0	⁸ 1
	01	¹ 1	⁵ 0	¹³ 0	⁹ X
	11	³ 0	⁷ 0	¹⁵ 0	¹¹ 0
	10	² 1	⁶ 0	¹⁴ 1	¹⁰ X

Reducing Canonical Expressions

1. Draw K-map
2. Identify Prime implicants

Are all the Prime implicants also essential?

A. Yes

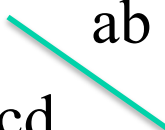
B. No



ab		00	01	11	10
cd	00	⁰ 1	⁴ 0	¹² 0	⁸ 1
	01	¹ 1	⁵ 0	¹³ 0	⁹ X
	11	³ 0	⁷ 0	¹⁵ 0	¹¹ 0
	10	² 1	⁶ 0	¹⁴ 1	¹⁰ X

Reducing Canonical Expressions

1. Prime implicants: $\Sigma m(0, 1, 8, 9)$, $\Sigma m(0, 2, 8, 10)$, $\Sigma m(10, 14)$
2. Essential Primes: $\Sigma m(0, 1, 8, 9)$, $\Sigma m(0, 2, 8, 10)$, $\Sigma m(10, 14)$



		ab			
		00	01	11	10
cd	00	0 1	4 0	12 0	8 1
	01	1 1	5 0	13 0	9 X
	11	3 0	7 0	15 0	11 0
	10	2 1	6 0	14 1	10 X

PI Q: Do the E-primes cover the entire on set?

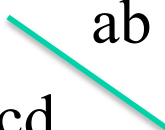
A. Yes

B. No

Reducing Canonical Expressions

1. Prime implicants: $\Sigma m(0, 1, 8, 9)$, $\Sigma m(0, 2, 8, 10)$, $\Sigma m(10, 14)$
2. Essential Primes: $\Sigma m(0, 1, 8, 9)$, $\Sigma m(0, 2, 8, 10)$, $\Sigma m(10, 14)$
3. Min exp: $\Sigma (\text{Essential Primes}) = \Sigma m(0, 1, 8, 9) + \Sigma m(0, 2, 8, 10) + \Sigma m(10, 14)$

$$f(a,b,c,d) = ?$$

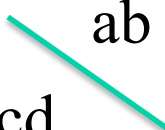


		ab			
		00	01	11	10
cd	00	0 1	4 0	12 0	8 1
	01	1 1	5 0	13 0	9 X
	11	3 0	7 0	15 0	11 0
	10	2 1	6 0	14 1	10 X

Reducing Canonical Expressions

1. Prime implicants: $\Sigma m(0, 1, 8, 9)$, $\Sigma m(0, 2, 8, 10)$, $\Sigma m(10, 14)$
2. Essential Primes: $\Sigma m(0, 1, 8, 9)$, $\Sigma m(0, 2, 8, 10)$, $\Sigma m(10, 14)$
3. Min exp: $\Sigma(\text{Essential Primes}) = \Sigma m(0, 1, 8, 9) + \Sigma m(0, 2, 8, 10) + \Sigma m(10, 14)$

$$f(a,b,c,d) = b'c' + b'd' + acd'$$



ab		00	01	11	10
cd	00	0 1	4 0	12 0	8 1
	01	1 1	5 0	13 0	9 X
	11	3 0	7 0	15 0	11 0
	10	2 1	6 0	14 1	10 X

Reading

[Harris] Chapter 2, 2.7