

Quiz 1 Solutions

1 Prove that for all $n \geq 0$, 3 divides $4^n + 5$.

Let us prove this by induction

Base Case

For $n = 0$, $4^n + 5 = 4^0 + 5 = 6$. And, 3 divides 6.

Induction Hypothesis

Assume, for some n , $4^n + 5$ is divisible by 3. This means, there is an integer q such that $4^n + 5 = 3q$.

Induction Step

Let us prove the statement for $n + 1$.

$$\begin{aligned} 4^{n+1} + 5 &= 4 \cdot 4^n + 5 \\ &= 3 \cdot 4^n + (4^n + 5) \\ &= 3 \cdot 4^n + 3q \\ &= 3 \cdot (4^n + q) \end{aligned}$$

Hence, $4^{n+1} + 5 = 3 \cdot q'$, where $q' = 4^n + q$, an integer. So, $4^{n+1} + 5$ is divisible by 3.

Note: There are other proofs which don't involve induction, but use modulo operations, which are valid too. ■

2 Let n be an integer. Let the remainder when n is divided by b is a . Prove that if n is written in the base b representation as

$$n = x_0 \cdot b^0 + x_1 \cdot b^1 + \dots + x_k \cdot b^k,$$

then x_0 must be equal to a .

As each x_i is a digit in base b representation, $x_i \leq b - 1$. Specifically, $x_0 \leq b - 1$. Now, n can be rewritten as:

$$\begin{aligned} n &= x_0 + x_1 \cdot b^1 + \dots + x_k \cdot b^k \\ &= x_0 + b \cdot (x_1 + \dots + x_k \cdot b^{k-1}) \end{aligned}$$

Hence, $n = x_0 + b \cdot q$ where q is some integer. As $b \cdot q$ is divisible by b and $x_0 \leq b - 1$, n leaves a remainder of x_0 when divided by b . ■

3 How many functions are there from the set of english alphabet $\{a, b, \dots, z\}^n$ to the set $\{0, 1, 2, \dots, 9\}$.

Domain is $\{a, b, \dots, z\}^n$. It's size is 26^n as it is an n dimensional cartesian product of a set of size 26. Size of range is 10.

Number of functions is 10^{26^n} as each element in domain has 10 options from range. ■