

Quiz 4 Solutions

1 Let $G = (v, E)$ be a complete undirected graph. If $s, t, w \in V$ what is the probability that a path of length k from s to t does go through w ?

Let n be the number of vertices in the graph.

Counting number of paths of length k from s to t :

This can be imagined as filling $k - 1$ slots between s and t by choosing vertices from the remaining $n - 2$ vertices (all vertices except s, t), without replacement. This can be done in $(n - 2) \times (n - 3) \times (n - 4) \times \dots \times (n - k)$ ways.

Counting number of paths of length k from s to t containing vertex w :

Similar to previous case, this can be thought of as filling $k - 1$ slots between s and t however, w should be one of them. So, choose a slot for w - this can be done in $k - 1$ ways. Then for remaining $k - 2$ slots, fill them using remaining $n - 3$ vertices (all vertices except s, t, w). This can be done in $(n - 3) \times (n - 4) \times \dots \times (n - k)$ ways. So, overall count is $(k - 1) \times (n - 3) \times (n - 4) \times \dots \times (n - k)$

Probability

So, the probability is: $\frac{(k-1) \times (n-3) \times (n-4) \times \dots \times (n-k)}{(n-2) \times (n-3) \times (n-4) \times \dots \times (n-k)} = \frac{k-1}{n-2}$

Bonus - Quick tricks to check if your answer is correct

When $k = 1$, probability should be zero because a path of length 1 cannot go via any other vertex w . When $n = 3, k = 2$, probability should be 1 because the graph will be a triangle containing three vertices s, w, t and any path of length 2 with s, t should go via w .

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2 Two unbiased dice are rolled and one unbiased coin is tossed. The nal outcome is $a \times b \times c$ where a, b, c, d are determined as follows:

- If number i is rolled by the rst dice then $a = i$.
- If number j is rolled by the second dice then $b = j$.
- If the outcome of the coin toss is HEAD then $c = 1$ else $c = 0$.

What is the expected outcome of the above experiment?

a, b are the numbers we get from first and second dice rolls respectively and c is 1 if we get head in a coin toss. Clearly, all the events a, b, c are mutually independent. So, we have:

$$E[a \times b \times c] = E[a] \times E[b] \times E[c]$$

Note: This formula is true only because the events are independent.

We know that $E[a] = E[b] = \sum_{i=1}^6 i \times \frac{1}{6} = \frac{7}{2}$.

And, $E[c] = 0 \times 0.5 + 1 \times 0.5 = 0.5$.

So, the final answer is:

$$\begin{aligned} E[a \times b \times c] &= E[a] \times E[b] \times E[c] \\ &= \frac{7}{2} \times \frac{7}{2} \times \frac{1}{2} \\ &= \frac{49}{8} \end{aligned}$$

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