

Quiz 2 Solutions

1 Count the number of 0/1-strings of length n that contains at least $n - 3$ consecutive 1s. Give clear reasoning for your arguments.

The answer is 20. Let (1^*) denote the largest consecutive set of 1's in a string. We want to count strings that have at least $n - 3$ consecutive 1's. They fall under one of these categories:

1. All 1's: 1 string.
2. One 0 and $n - 1$ 1's. These are different possibilities for zero to be present: $0(1^*)$, $10(1^*)$, $110(1^*)$, $(1^*)011$, $(1^*)01$, $(1^*)0$
3. Two zeros and $n - 2$ 1's: Nine options. $00(1^*)$, $010(1^*)$, $100(1^*)$, $(1^*)00$, $(1^*)010$, $(1^*)001$, $0(1^*)0$, $10(1^*)0$, $0(1^*)01$
4. Three zeroes: Three options. $000(1^*)$, $00(1^*)0$, $0(1^*)00$, $(1^*)000$

Answer is $1+6+9+2=20$.

[Credits: This answer is based on student's answer given in post #98 of Piazza (by Harvey Yu)] ■

2 How many 9 digit positive integers are there such that:

1. the first digit is not 0 and all the digits are distinct
2. the first digit is not 0 and no two consecutive digits are same.
3. the first digit is not 0 and the 9 digits must contain at least one 6 and at least one 9.

1. $9 \times 9!$. First digit has 9 options between 1-9. The remaining 8 digits need to be selected from 9 other numbers and have to be distinct. This can be done in $9 \times 8 \times \dots \times 2$.

2. 9^9 . First digit has 9 options. Every other digit cannot be same as previous digit. Hence each have 9 options.
3. This needs to be solved using inclusion-exclusion. Consider the following sets:
 - U = Set of 9 digit numbers with first digit is non-zero. $|U| = 9 \times 10^8$.
 - A = Set of 9 digit numbers with first digit is not zero and number does not contain 6. $|A| = 8 \times 9^8$.
 - B = Set of 9 digit numbers with first digit is not zero and the number does not contain 9. $|B| = 8 \times 9^8$.
 - $A \cap B$ = Set of 9 digit numbers with first digit is not zero and the number does not contain 6 or 9. $|A \cap B| = 7 \times 8^8$.

We need to compute $|U - (A \cup B)|$. And this is, by inclusion exclusion, $|U| - |A| - |B| + |A \cap B| = 9 \times 10^8 - 2 \times 8 \times 9^8 + 7 \times 8^8$

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