

Assignment 6 Solutions

1

- (a) How many functions are there from $\{1, \dots, n\}$ to $\{1, \dots, k\}$?
- (b) A function is said to be increasing if $a > b$ implies $f(a) > f(b)$. How many increasing functions are there from $\{1, \dots, n\}$ to $\{1, \dots, k\}$?
- (b) A function is said to be non-decreasing if $a > b$ implies $f(a) \geq f(b)$. How many non-decreasing functions are there from $\{1, \dots, n\}$ to $\{1, \dots, k\}$?

Part (a):

There are k^n functions because each element in domain has k choices among range values and there are n elements in domain.

Part(b):

Notice the following properties:

1. No two elements from domain can be mapped to same element in range.
2. If $k < n$, we cannot have a valid function. So, $k \geq n$.
3. If $k = n$, then there is exactly one function that maps domain to range.

From property 1 mentioned above, there are exactly n elements in range that are mapped by any increasing function and remaining $k - n$ elements are left out. And, from property 3, once we determine these n elements from range, there is exactly one function that maps domain to these n elements. Hence, the number of functions is same as number of ways of choosing n elements out of k from range. This is $\binom{k}{n}$.

Part (c):

This is same as distributing n indistinguishable items to k people, that is, $\binom{n+k-1}{n}$. The n elements in domain, arranged in increasing order, can be imagined as balls and the k elements in range can be imagined as walls. This makes the two problems identical. ■

2 How many 8 digit positive integers are there such that the rst digit is not 0 and the integer is even and one of the 8 digits is 6.

Let us list the conditions and find corresponding constraints on counting:

1. **First digit is not 0:** This means there are only 9 choices for first digit.
2. **The integer is even:** So, the last digit should be one of 0, 2, 4, 6, 8. Hence, there are 5 choices for last digit.
3. **At least one of the digits is 6:** We shall handle this using complements.

Complement of condition 3 is "*none of the digits are 6*". Hence, for our solution, we shall compute the cases satisfying only condition 1 and 2 and the subtract the cases satisfying condition 1, 2 and complement of 3.

Number of numbers satisfying condition 1 and 2

First digit has 9 options and last digit has 5 options. Remaining 6 digits have 10 options each. So, answer is $9 \times 10^6 \times 5 = 45 \times 10^6$.

Number of numbers satisfying condition 1, 2 and complement of 3

Now, since the digit 6 cannot appear anywhere (due to complement of condition 3), first digit has 8 options and last digit has 4 options. Remaining 6 digits have 9 options each. So, answer is $8 \times 9^6 \times 5 = 32 \times 9^6$.

So, the final answer is $45 \times 10^6 - 32 \times 9^6$. ■

3 Give the closed form expression for $\sum_{k=0}^n \binom{n}{k} (2/3)^k$.

Recall the binomial formula:

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} (x)^k$$

The summation in the question resembles RHS of binomial formula with $x = 2/3$. So, the answer is:

$$\left(1 + \frac{2}{3}\right)^n = \left(\frac{5}{3}\right)^n$$

■

4 Solve the following recurrences: (assume $T(0) = T(1) = 1$)

(a) $T(n) = 2T(n/2) + 5$. (Assume n is a power of 2).

(b) $T(n) = T(n-1) + 15$

(c) $T(n) = T(n/2) + n$ (Assume n is a power of 2).

(d) $T(n) = T(n-1) + n$

(e) $T(n) = 2T(n-1)$

Part (a)

Since we can assume n is a power of 2, let $n = 2^k$ for some k . To unroll the recurrence relation,

$$T(n) = 2T(n/2) + 5$$

since we need to substitute for $T(n/2)$, replace n with $n/2$ and $n/4$ in the above equation:

$$T(n/2) = 2T(n/4) + 5 \tag{1}$$

$$T(n/4) = 2T(n/8) + 5 \tag{2}$$

Hence, we get:

$$\begin{aligned} T(n) &= 2T(n/2) + 5 \\ &= 2 \times (2T(n/4) + 5) + 5 \text{ (from eqn (1))} \\ &= 4 \times T(n/4) + 2 \times 5 + 5 \\ &= 4 \times (2T(n/8) + 5) + 2 \times 5 + 5 \text{ (from eqn (2))} \\ &= 8 \times T(n/8) + 4 \times 5 + 2 \times 5 + 5 \end{aligned}$$

And, continuing the pattern, after k steps, we get,

$$\begin{aligned}
 T(n) &= 2^k \times T(n/2^k) + 2^{k-1} \times 5 + 2^{k-2} \times 5 + \dots + 2^0 \times 5 \\
 &= 2^k \times T(1) + [2^{k-1} + \dots + 1] \times 5 \text{ (because } n = 2^k) \\
 &= 2^k \times T(1) + [2^k - 1] \times 5 \text{ (sum of Geometric Progression)} \\
 &= 6 \times 2^k - 5 \text{ (because } T(n) = 1) \\
 &= 6n - 5 \text{ (because } n = 2^k)
 \end{aligned}$$

Now, let us verify the solution obtained. $T(1) = 6 \times 1 = 5$.

For recurrence relation, $T(n/2) = 6 \times (n/2) - 5 = 3n - 5$. Now, from question, we have:

$$\begin{aligned}
 T(n) &= 2T(n/2) + 5 \\
 &= 2(3n - 5) + 5 \\
 &= 6n - 5
 \end{aligned}$$

And, this verifies the solution. ■

Part (b)

We have

$$T(n) = T(n - 1) + 15$$

In the above equation, by replacing n with $n - 1, n - 2$ and so on, we get:

$$\begin{aligned}
 T(n) &= T(n - 1) + 15 \\
 &= (T(n - 2) + 15) + 15 \\
 &= (T(n - 3) + 15) + 15 + 15 \\
 &= (T(n - 4) + 15) + 15 + 15 + 15
 \end{aligned}$$

Finally, we get,

$$T(n) = T(0) + 15n = 15n + 1$$

To verify the solution, for $n = 1$, $T(0) = 15 \times 0 + 1 = 1$. Now, assuming $T(n - 1) = 15(n - 1) + 1$, we get:

$$\begin{aligned} T(n) &= T(n - 1) + 15 \\ &= 15(n - 1) + 1 + 15 \\ &= 15n + 1 \end{aligned}$$

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Part (c)

$$T(n) = T(n/2) + n$$

This question is very similar to Part (a). Assume $n = 2^k$ for some k . Following the same pattern as Part (a), we have:

$$\begin{aligned} T(n) &= T(n/2) + n \\ &= T(n/4) + n/2 + n \\ &= T(n/8) + n/4 + n/2 + n \end{aligned}$$

Continuing for k steps, we get:

$$\begin{aligned} T(n) &= T(n/2^k) + n/2^{k-1} + \dots + n/4 + n/2 + n \\ &= 1 + 2 + 2^2 + \dots + 2^{k-1} + 2^k \text{ (because } n = 2^k \text{)} \\ &= 2^{k+1} - 1 \\ &= 2n - 1 \end{aligned}$$

To verify, $T(1) = 2 \times 1 - 1 = 1$. Now, assume $T(n/2) = 2(n/2) - 1$. From recurrence relation, we have:

$$\begin{aligned} T(n) &= T(n/2) + n \\ &= 2(n/2) - 1 + n \\ &= 2n - 1 \end{aligned}$$

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Part (d)

$$T(n) = T(n-1) + n$$

This was solved in class. This is essentially, sum of first n integers. The answer is

$$T(n) = \frac{n(n+1)}{2}$$

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Part (e)

$$T(n) = 2T(n-1)$$

For this, let us use the same method to unroll the recurrence.

$$\begin{aligned} T(n) &= 2 \times T(n-1) \\ &= 2(2 \times T(n-2)) \\ &= 2 \times 2 \times (2 \times T(n-3)) \end{aligned}$$

Finally, we get:

$$\begin{aligned} T(n) &= 2^n \times T(0) \\ &= 2^n \end{aligned}$$

Now, let us verify $T(n) = 2^n$. For $n = 0$, $T(0) = 2^0 = 1$. Assume $T(n-1) = 2^{n-1}$. From recurrence, we get:

$$\begin{aligned} T(n) &= 2 \times T(n-1) \\ &= 2 \times 2^{n-1} \\ &= 2^n \end{aligned}$$

This verifies the solution.

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