

Assignment 3 Solutions
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1 The triangle inequality says that for any two real numbers x and y ,

$$|x + y| \leq |x| + |y|$$

Show that for any n real numbers $x_1, x_2, \dots, x_n$ we have,

$$|x_1 + \dots + x_n| \leq |x_1| + |x_2| + \dots + |x_n|.$$

Let us prove this by induction on n

Base Case

For $n = 2$, the hypothesis is:

$$|x_1 + x_2| \leq |x_1| + |x_2|$$

This is straightforward from the triangle inequality by taking $x = x_1, y = x_2$.

Induction Hypothesis

Assume that the statement is valid for some k , ie,

$$|x_1 + \dots + x_k| \leq |x_1| + |x_2| + \dots + |x_k|$$

Induction Step

Now, let us prove the statement for $k + 1$.

$$|x_1 + \dots + x_{k+1}| = |(x_1 + \dots + x_k) + x_{k+1}|$$

Now, use triangle inequality with $x = x_1 + \dots + x_k$ and $y = x_{k+1}$. This gives us:

$$\begin{aligned} |x + y| &\leq |x| + |y| \\ |(x_1 + \dots + x_k) + x_{k+1}| &\leq |x_1 + \dots + x_k| + |x_{k+1}| \\ &\leq |x_1| + |x_2| + \dots + |x_k| + |x_{k+1}| \text{ (from induction hypothesis)} \end{aligned}$$

This proves the statement for $k + 1$. ■

2 If every two cities in state S are joined by a one-way road, then it is possible to find a starting city A and a route from A that passes through every city exactly once.

The question says that between any two cities X and Y , either there's an one-way road from X to Y or from Y to X , but not both. Then show that there is a path from some city A which visits all the cities exactly once.

This problem is very similar to the Football league problem #7 in Homework 1. This problem can be modelled as a directed graph in which cities are represented as vertices and there's an edge from vertices X to Y if there's a road from cities X to Y . And then, our goal is to find a path in this graph which visits every vertex in the graph.

The rest of the solution is very similar to the argument in Problem 7 of homework 1. We use induction of number of cities, say n .

As a base case, if $n = 2$, there are only two cities X and Y and the path is simply the edge between X and Y .

For induction hypothesis, assume that the statement is true for any set of n cities. That is, for any set of n cities, we can find a path which visits all the cities exactly once.

Let's prove the statement for $n + 1$ cities. Leave out $n + 1^{th}$ city (say, C_{n+1}) and for the remaining n cities, by induction hypothesis, we can find a path which visits all these n cities exactly once. Let us label the cities such that this path is $C_1 \rightarrow C_2 \rightarrow C_3 \dots \rightarrow C_n$. Now, if all roads are going towards C_{n+1} , then add C_{n+1} at the end of this path. Otherwise, let C_i be the first city in this path such that there is a road from C_{i-1} to C_{n+1} and from C_{n+1} to C_i . Place C_{n+1} between C_{i-1} and C_i to get the desired path. ■

3 In a party there are n people $p_1, p_2, \dots, p_n$. Some of them shake hand with each other. So at the end the person p_1 shakes hand with d_1 number of other people in the party. Similarly p_2 shakes hand with d_2 number of other people in the party. Likewise for all $1 \leq i \leq n$ the person p_i shakes hand with d_i number of other people in the party. At the end the total number of handshakes done is m . Show that

$$2m = (d_1 + d_2 + \dots + d_n)$$

Every handshake that happens between two people increases the handshake count of both people by 1. Hence, sum of handshake counts of all the people is twice the number of handshakes. We can give a formal proof using induction on number of handshakes, m .

Base Case:

When $m = 1$, that is with only one handshake, say between p_i and p_j , then $d_i = d_j = 1$ and number of handshakes is zero for others. So, we get $(d_1 + d_2 + \dots + d_n) = d_i + d_j + 0 = 2 = 2 * 1 = 2m$.

Induction Hypothesis:

Let us assume that the statement is true for m handshakes. That is, after m handshakes, let $d_1, d_2, \dots, d_n$ denote the number of handshakes. Then, we have:

$$2m = (d_1 + d_2 + \dots + d_n)$$

Induction Step:

Let us look at $m + 1^{th}$ handshake between some p_i and p_j . Total number of handshakes is now $m + 1$ and after this handshake, p_i 's handshake count becomes $d_i + 1$ and p_j 's becomes $d_j + 1$. Hence, total number of handshakes now is:

$$\begin{aligned} & d_1 + d_2 + \dots + (d_i + 1) + \dots + (d_j + 1) + \dots + d_n \\ &= d_1 + d_2 + \dots + d_i + \dots + d_j + \dots + d_n + 2 \\ &= 2m + 2 \text{ (From induction hypothesis)} \\ &= 2(m + 1) \end{aligned}$$

■

4 How many edges are there in an undirected graph with 1000 vertices?

In such questions, it is better to start with smaller number of vertices and then generalize it to any number n . Let us analyze this problem for any graph of n vertices, and find out how many edges are possible in this graph.

There are multiple ways to solve this problem. Here are two of them:

1. Let us number the vertices $v_1, v_2, \dots, v_n$. v_1 can be connected to remaining $n - 1$ vertices using $n - 1$ edges. And for v_2 , since we have

already connected v_1, v_2 , we can connect v_2 with remaining $n - 2$ vertices using $n - 2$ edges and so on. This way, vertex v_i can be connected to all v_j s for $j > i$ with $n - i$ edges. Total number of edges is $n - 1 + n - 2 + n - 3 + \dots + 1$. This is $\frac{n(n-1)}{2}$.

2. Notice that each edge has two end points. So, to make an edge, we just need to choose two end points out of n . Total number of edges is total number of ways in which you can choose two points out of n for forming the edge. This is given by $\binom{n}{2} = \frac{n(n-1)}{2}$.

And hence, for this particular problem with $n = 1000$, the number of possible edges is $\frac{999 \times 1000}{2}$ ■

5 How many distinct kind of graphs on 10 vertices are there?

Again, we shall solve this for a graph of n vertices. We say two graphs of n vertices are different when they have different set of edges. Hence, the question can be rephrased as, how many possible edge sets can be formed for a graph of n vertices.

We know that set of all possible edges (let's denote it by E) is $\binom{n}{2}$ and the number of possible edge sets is number of ways in which we can form subsets of E .

Since $|E| = \binom{n}{2}$, number of possible subsets is $2^{\binom{n}{2}}$. Hence, number of graphs is $2^{\binom{n}{2}}$. In particular, for $n = 10$, the answer is 2^{45} . ■