

Assignment 2 Solutions  
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1 Prove that

$$\sum_{i=0}^{k-1} ab^i = a \frac{b^k - 1}{b - 1}$$

Let us prove this by induction on  $k$

**Base Case**

For  $k = 1$ , the term on the left hand side (LHS) has only one term,  $ab^0 = a$ .  
And, RHS evaluates to  $a \frac{b-1}{b-1} = a$ .

**Induction Hypothesis**

Assume that the statement is valid for some  $k$ , ie,

$$\sum_{i=0}^{k-1} ab^i = a \frac{b^k - 1}{b - 1}$$

**Induction Step**

Now, let us prove the statement for  $k + 1$ .

$$\begin{aligned} \sum_{i=0}^k ab^i &= \sum_{i=0}^{k-1} ab^i + ab^k \\ &= a \frac{b^k - 1}{b - 1} + ab^k \text{ (from induction hypothesis)} \\ &= a \frac{b^k - 1 + b^{k+1} - b^k}{b - 1} \\ &= a \frac{b^{k+1} - 1}{b - 1} \end{aligned}$$

This proves the statement for  $k + 1$ . ■

**2** Prove that

$$\sum_{i=0}^k (a + ib) = \frac{2a + kb}{2}(k + 1)$$

Let us prove this by induction on  $k$

**Base Case**

For  $k = 0$ , LHS has only one term,  $a$ . And RHS is  $\frac{2a+0}{2}(0 + 1) = a$ .

**Induction Hypothesis**

Assume that the statement is valid for some  $k$ , ie,

$$\sum_{i=0}^k (a + ib) = \frac{2a + kb}{2}(k + 1)$$

**Induction Step**

Now, let us prove the statement for  $k + 1$ .

$$\begin{aligned}\sum_{i=0}^{k+1} (a + ib) &= \sum_{i=0}^k (a + ib) + [a + (k + 1)b] \\ &= \frac{2a + kb}{2}(k + 1) + [a + (k + 1)b] \text{ (from induction hypothesis)} \\ &= \frac{2a(k + 1) + kb(k + 1) + [2a + (k + 1)2b]}{2} \\ &= \frac{2a(k + 1) + 2a + kb(k + 1) + (k + 1)2b}{2} \\ &= \frac{2a(k + 2) + b(k + 1)(k + 2)}{2} \\ &= \frac{2a + (k + 1)b}{2}(k + 2)\end{aligned}$$

■

**3** Let  $n > 1$  be an integer. In a football league there are  $n$  teams. Every two teams have played against each other exactly once, and in match no draw is allowed. Prove that it is possible to number the teams in such a way that team  $i$  beats  $(i + 1)$  for  $i = 1, 2, \dots, n - 1$

Use induction on  $n$ .

### Base Case

For  $n = 2$ , there are only two teams. Whoever won the match between the two will be numbered 1 and the other team is numbered 2.

### Induction Hypothesis

Assume that the statement is true for  $n$ . That is, if there are  $n$  teams, we can assign numbers 1 to  $n$  so that team  $i$  has won against team  $i + 1$ .

### Induction Step

Now, let us prove the statement for  $n + 1$ . Take a tournament of  $n + 1$  teams. Out of the  $n + 1$  teams, consider a subset of  $n$  teams first. By induction hypothesis, we can number these  $n$  teams 1 to  $n$  such that team  $i + 1$  has lost to team  $i$ . Now we shall "fit in" the remaining  $n + 1^{\text{th}}$  team in this order.

If this  $n + 1^{\text{th}}$  team has lost to all  $n$  teams, then assign a number  $n + 1$ . Since it has lost to team  $n$ , the order is preserved.

Otherwise, look at the first team in the order, say team with number  $j (\leq n)$ , it has won against. Assign number  $j$  to this  $n + 1^{\text{th}}$  team and increment the number of all the teams that are on or after  $j$  in the order by 1. ■

4 Prove that  $2002^{n+2} + 2003^{2n+1}$  is divisible by 4005.

The statement is incorrect! Here's a quick way to check the same. If a number is a multiple of 4005, then it should also be a multiple of 5 (as 5 divides 4005). So, is  $2002^{n+2} + 2003^{2n+1}$  a multiple of 5? Let us check the remainder of individual terms when divided by 5.

$2002^{n+2}$  can be written as  $(2000 + 2)^{n+2}$ . As 5 divides 2000, remainder of  $(2000 + 2)^{n+2}$  is same as remainder of  $2^{n+2}$  when divided by 5. Similarly,  $2003^{2n+1}$  is equivalent to  $3^{2n+1}$  for remainder when divided by 5. Now the sum boils down to  $2^{n+2} + 3^{2n+1} = 4 \cdot 2^n + 3 \cdot 9^n$ . Clearly, when  $n = 2$ , this sum is  $4 \cdot 2^2 + 3 \cdot 9^2 = 16 + 243 = 259$  which is not divisible by 5. ■

5 The Fibonacci sequence is dened as  $x_0 = 0, x_1 = 1$  and  $x_{n+2} = x_{n+1} + x_n$  for all non-negative integers  $n$ . Prove that

(a)  $x_m = x_{r+1}x_{m-r} + x_r x_{m-r-1}$  for all integers  $m \geq 1$  and  $0 \leq r \leq m-1$ ;

(b)  $x_d$  divides  $x_{kd}$  for all positive integers  $d$  and  $k$ .

**Part (a)**

Let us prove this by induction on  $r$  in the range  $0 \leq r \leq m-1$ .

**Base Case**

For  $r = 0$ ,  $x_{r+1}x_{m-r} + x_r x_{m-r-1} = x_1 x_m + x_0 x_{m-1} = 1x_m + 0x_{m-1} = x_m$ .

**Induction Hypothesis**

Now, assume for some  $r$  in range 0 to  $m-2$  that  $x_m = x_{r+1}x_{m-r} + x_r x_{m-r-1}$ .

**Induction Step**

Take  $r+1$  with  $r+1 \leq m-1$ . Now, with  $r+1$ ,

$$\begin{aligned} RHS &= x_{r+2}x_{m-r-1} + x_{r+1}x_{m-r-2} \\ &= (x_{r+1} + x_r)x_{m-r-1} + x_{r+1}x_{m-r-2} \\ &= x_{r+1}x_{m-r-1} + x_r x_{m-r-1} + x_{r+1}x_{m-r-2} \\ &= x_{r+1}(x_{m-r-1} + x_{m-r-2}) + x_r x_{m-r-1} \\ &= x_{r+1}x_{m-r} + x_r x_{m-r-1} \\ &= x_m \text{ (from induction hypothesis)} \end{aligned}$$

Note that here  $r+1 \leq m-1$  is important because otherwise  $x_{m-r-2}$  will not be valid.

**Part (b)**

Use induction on  $k$ .

**Base Case**

For  $k = 1$ ,  $x_{kd} = x_d$  and  $x_d$  trivially divides  $x_{kd}$ .

**Induction Hypothesis**

Assume for some  $k$ ,  $x_d$  divides  $x_{kd}$ . That is,  $x_{kd} = q \cdot x_d$  for some integer  $q$ .

**Induction Step**

Now, consider  $x_{(k+1)d}$ . By using  $m = (k+1)d$  and  $r = d$  in the identity in

part (a), we get:

$$\begin{aligned}
 x_{(k+1)d} &= x_{d+1}x_{(k+1)d-d} + x_d x_{(k+1)d-d-1} \\
 &= x_{d+1}x_{kd} + x_d x_{kd-1} \\
 &= q \cdot x_{d+1}x_d + x_d x_{kd-1} \\
 &= x_d(qx_{d+1} + x_{kd-1})
 \end{aligned}$$

■

**6** (Hard Problem) For natural numbers  $p$  and  $q$ , the Ramsey number  $R(p, q)$  is dened as the smallest integer  $n$  so that among any  $n$  people, there exist  $p$  of them who know each other, or there exist  $q$  of them who dont know each other. Note that  $R(p, 1) = R(1, q) = 1$ . Prove that:

- (a)  $R(p + 1, q + 1) \leq R(p, q + 1) + R(p + 1, q)$
- (b)  $R(p, q) \leq C_{p-1}^{p+q-2}$

In both the parts, we need induction on both  $p$  and  $q$ . A rigorous mathematical proof would involve taking an arbitrary, but fixed value for  $q$  (say,  $q = q_0$ ) and then inducting over  $p$ . Once we have proved it for all  $p$  and fixed  $q$ , we can use induction over  $q$ . Here, we give a sketch of proof of induction step, which is the most critical part.

### Part (a)

As induction hypothesis, assume the following two statements:

- $R(p, q + 1)$ : In a group of  $R(p, q + 1)$  people there are either  $p$  people who know each other or  $q + 1$  people who don't know each other.
- $R(p + 1, q)$ : In a group of  $R(p + 1, q)$  people there are either  $p + 1$  people who know each other or  $q$  people who don't know each other.

Now, let us prove the following statement: In a group of  $R(p, q + 1) + R(p + 1, q)$ , there are either  $p + 1$  people who know each other or  $q + 1$  people who don't know each other.

Take any group  $G$  of  $R(p, q + 1) + R(p + 1, q)$  people and select a random person  $x$ . Let  $F \subseteq G$  be the set of people  $x$  knows and  $E \subseteq G$  be the set of people  $x$  doesn't know.

Now, either  $|F| \geq R(p, q + 1)$  or  $|E| \geq R(p + 1, q)$ . Otherwise, if  $|F| \leq R(p, q + 1) - 1$  and  $|E| \leq R(p + 1, q) - 1$ , then  $|G| = |F| + |E| + 1 \leq R(p, q + 1) + R(p + 1, q) - 1$ , a contradiction.

**Case 1:**  $|F| \geq R(p, q + 1)$

By induction hypothesis,  $F$  contains either  $q + 1$  people who don't know each other, or  $p$  people who know each other. If  $F$  contains  $q + 1$  people who don't know each other, then  $G$  also contains the same set of people and we are done. Otherwise,  $F$  contains  $p$  people who know each other. And,  $x$  knows everyone from  $F$  and especially these  $p$  people. So, along with  $x$ , we have found a subset of  $p + 1$  people who know each other.

**Case 2:**  $|E| \geq R(p + 1, 1)$

The argument here is symmetric to Case 1. By induction hypothesis,  $E$  contains either  $p + 1$  people who know each other, or  $q$  people who don't know each other. If  $E$  contains  $p + 1$  people who know each other, we are done. Otherwise,  $E$  contains  $q$  people who don't know each other. And,  $x$  doesn't know anyone from  $E$  and especially these  $q$  people. So, along with  $x$ , we have found a subset of  $q + 1$  people who don't know each other.

This concludes the proof of part(a). Note that the problem can be visualized as a graph problem where people represent vertices and if two people know each other we add an edge between corresponding vertices.

## Part (b)

Again, we use induction on both  $p$  and  $q$ . By induction hypothesis, assume that:

1.  $R(p - 1, q) \leq C_{p-2}^{p+q-3}$
2.  $R(p, q - 1) \leq C_{p-1}^{p+q-3}$

So, we have:

$$\begin{aligned}
 R(p, q) &\leq R(p, q - 1) + R(p - 1, q) \text{ (From part (a))} \\
 &\leq C_{p-1}^{p+q-3} + C_{p-2}^{p+q-3} \text{ (From induction hypothesis)} \\
 &= C_{p-1}^{p+q-2}
 \end{aligned}$$

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