

CSE 21: Assignment Set 1

1. Prove that there is a unique representation of any positive integer in any base b . State the problem in mathematical terms and then prove it.

Answer: Let us prove by contradiction. Let N be the integer for which there are two different representations in base b . Let the following be the two representations

$$N = x_0b^0 + x_1b^1 + \dots + x_kb^k, \quad (1)$$

$$N = y_0b^0 + y_1b^1 + \dots + y_\ell b^\ell, \quad (2)$$

Since by assumption the two representations are different so there must be an i such that $x_i \neq y_i$. Let ℓ be the largest such i . So for all $i > \ell$ we have $x_i = y_i$.

Without loss of generality let $y_\ell > x_\ell$

Now consider Equation 1 - Equation 2. We get

$$0 = (x_0 - y_0)b^0 + \dots + (x_\ell - y_\ell)b^\ell.$$

Or if we move the last term to the other side

$$(y_\ell - x_\ell)b^\ell = (x_0 - y_0)b^0 + \dots + (x_{\ell-1} - y_{\ell-1})b^{\ell-1}. \quad (3)$$

Since $y_\ell > x_\ell$ so $(y_\ell - x_\ell)b^\ell \geq b^\ell$. Thus the LHS of the above equation is $\geq b^\ell$. On the other hand for all $i < \ell$, $x_i - y_i \leq (b-1)$. So the RHS is at most

$$(b-1)(b^0 + \dots + b^{\ell-1}) = b^\ell - 1$$

The last equality is from the sum of geometric series.

Note: We use the formula about the sum of terms in a geometric progression. So if we have a series of the form

$$a, ab, ab^2, ab^3, \dots, ab^{k-1}$$

then the sum of all the k terms in the series is $a \frac{b^k - 1}{b - 1}$.

So in Equation 3 the LHS is at least b^ℓ while the RHS is at most $b^\ell - 1$. Thus a contradiction and hence our initial assumption that there are two representations cannot be correct.

2. What is the maximum integer that can be represented in base 2 using only 10 bits (that is, what is the largest integer which when represented in base 2 has at most length 10 representation).

answer: The maximum integer that can be represented by 10 bits in base 2 is $[1111111111]_2$ which is $(2^0 + 2^1 + \dots + 2^9) = 2^{10} - 1 = 1023$.

3. Write down the truth table of the function $f : \{0, 1\}^3 \rightarrow \{0, 1\}$, where

$$f(x_1, x_2, x_3) = (x_1 \vee x_2) \wedge x_3.$$

Answer: The truth table of the function is given below:

x_1	x_2	x_3	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

4. Let $x > -1$ be a real number. Prove that $(1 + x)^n \geq 1 + nx$ for all natural numbers n .

Answer: Let us prove by induction.

Base Case: ($n = 1$): In this case $(1 + x)^n = (1 + x) = 1 + nx$. So base case is true.

Now let induction hypothesis assume that for all $i < n$ $(1 + x)^i \geq 1 + ix$. In particular

$$(1 + x)^{n-1} \geq 1 + (n-1)x \tag{4}$$

Now let us multiple Equation 4 on both sides with $(1 + x)$. Since $x > -1$ so $(1 + x) > 0$. Thus the inequality does not change as we are multiplying both sides with a positive number. So we have

$$(1 + x)^n \geq (1 + x)(1 + (n-1)x)$$

The RHS is equal to $(1 + nx + (n-1)x^2)$ which is $\geq (1 + nx)$.

So by induction hypothesis we have $(1 + x)^n \geq 1 + nx$ for all n when $x > -1$.

5. Prove that $\sum_{i=1}^n i \times i! = (n+1)! - 1$.

Answer: This also follows from induction. The base case is $n = 1$. And the result follows from simple induction.

6. Let $\{a_n\}$ be a sequence of natural numbers such that $a_1 = 5$, $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all natural numbers n . Prove that $a_n = 2^n + 3^n$ for all natural number n .

Answer: This also follows from induction. The base case is $n = 1$ and $n = 2$. It is not hard to verify that the base cases are correct.

Now let us assume by induction hypothesis that for all $i < n + 2$, $a_i = 2^i + 3^i$. In particular $a_{n+1} = 2^{n+1} + 3^{n+1}$ and $a_n = 2^n + 3^n$.

Now since $a_{n+2} = 5a_{n+1} - 6a_n$ so by putting the values of a_n and a_{n+1} we have

$$a_{n+2} = 5(2^{n+1} + 3^{n+1}) - 6(2^n + 3^n) = 5(2^{n+1} + 3^{n+1}) - 3(2^{n+1}) - 2(3^{n+1})$$

It is easily seen that the RHS gives $(2^{n+2} + 3^{n+2})$. So by induction we prove the statement.

7. Let $n > 1$ be an integer. In a football league there are n teams. Every two teams have played against each other exactly once, and in match no draw is allowed. Prove that it is possible to number the teams in such a way that team i beats $(i + 1)$ for $i = 1, 2, \dots, n - 1$.

Answer: This is also proved by induction. The best way is to visualize it as a graph and then solve it there. We will discuss it in class.