

CSE 21: Practice Midterm

Maximum marks: 100
Total Time: 50 minutes

1. (15 Marks) How many ways can you divide n identical chocolates among k children such that each gets at least 2?

Answer: First we give 2 chocolates to every children. Thus we are left with $n - 2k$ chocolates. We then divide the chocolates among k children in anyway (where a child may not even get a chocolate). Note that since the children are all distinguishable so we can do it $\binom{n-2k+k-1}{k-1} = \binom{n-k-1}{k-1}$ ways.

2. (20 Marks) How many functions f are there from $\{1, \dots, n\}$ to $\{-1, 0, 1\}$ such that there exists at least one $i \in \{1, \dots, n\}$ where $f(i) = 0$.

Answer: The total number of functions from $\{1, \dots, n\}$ to $\{-1, 0, 1\}$ is 3^n . The total number of functions such that there is no i where $f(i) = 0$ is same as the the total number of functions from $\{1, \dots, n\}$ to $\{-1, 1\}$ which is 2^n . So the total number of functions from $\{1, \dots, n\}$ to $\{-1, 0, 1\}$ such that there exists at least one $i \in \{1, \dots, n\}$ where $f(i) = 0$ is $3^n - 2^n$.

3. (20 Marks) If $T(n) = T(n/3) + 15$ and $T(1) = 0$ then guess the expression of $T(n)$. (Assume that $T(n)$ is a power of 3.)

Answer: $T(n) = T(n/3) + 15$. Now opening up $T(n/3)$ we get:

$$T(n) = (T(n/9) + 15) + 15 = T(n/3^2) + 2 * 15$$

Opening up $T(n/9)$ we get

$$T(n) = T(n/3^3) + 3 * 15.$$

Thus we can guess that if we continue like this we will have

$$T(n) = T(n/3^k) + k * 15. \tag{1}$$

Now let $3^k = n$ that is $k = \log_3 n$. Plugging it in Equation 1 we have

$$T(n) = T(1) + 15 \log_3 n = 15 \log_3 n,$$

the last equality because $T(1) = 0$.

4. (15 Marks) Give a closed form expression of $1 + \binom{n}{1}2 + \binom{n}{2}4 + \cdots + \binom{n}{n}2^n$.

Answer: By binomial theorem we can see that the above expression is same as $(1 + 2)^n$ which is 3^n .

5. (15 Marks) If $T(n) = T(n-1) + n$ prove that $T(n) = n(n+1)/2$.

Note: here I forgot to give that $T(0) = 0$.

Answer: We prove this by induction on n .

Base Case: $n = 0$. Clearly $T(0) = 0 = n(n+1)/2$.

Induction Hypothesis: for all $n < k$ let $T(n) = n(n+1)/2$.

Inductive Step: Now consider the case of $n = k+1$. By the given recurrence we have $T(k+1) = T(k) + (k+1)$. By induction hypothesis $T(k) = k(k+1)/2$. So we have

$$T(k+1) = k(k+1)/2 + (k+1) = (k+1)(k+2)/2 = (k+1)((k+1)+1)/2.$$

So we prove that for all $n \geq 0$, $T(n) = n(n+1)/2$.

6. (15 Marks) Prove the following statement: $2^2 + 5^2 + 8^2 + \dots + (3n-1)^2 = \frac{1}{2}n(6n^2 + 3n - 1)$

Answer: We prove this by induction on n .

Let $T(n) = 2^2 + \dots + (3n-1)^2$. So to prove: $T(n) = \frac{1}{2}n(6n^2 + 3n - 1)$

Base Case: $n = 0$. Clearly $T(1) = 4$ and $\frac{1}{2}1(6 \cdot 1^2 + 3 \cdot 1 - 1) = 4$. Thus base case is proved.

Induction Hypothesis: for all $n < k$ let $T(n) = \frac{1}{2}n(6n^2 + 3n - 1)$.

Inductive Step: Now consider the case of $n = k+1$. By the given recurrence we have

$$T(k+1) = T(k) + (3(k+1)-1)^2 = T(k) + (9k^2 + 12k + 4).$$

By induction hypothesis $T(k) = \frac{1}{2}k(6k^2 + 3k - 1)$. So we have

$$T(k+1) = \frac{1}{2}k(6k^2 + 3k - 1) + (9k^2 + 12k + 4).$$

If we solve the last expression we see it is same as $\frac{1}{2}(k+1)(6(k+1)^2 + 3(k+1) - 1)$.

So we prove that for all $n \geq 1$, $T(n) = \frac{1}{2}n(6n^2 + 3n - 1)$.

7. (15 Marks) Prove that for all integer $n \geq 3$, $n^2 - 7n + 12 \geq 0$.

Answer: We prove this by induction on n .

Base Case: $n = 3$. Clearly $n^2 - 7n + 12 = 0 \geq 0$.

Induction Hypothesis: for all $n < k$ let $n^2 - 7n + 12 \geq 0$.

Inductive Step: Now consider the case of $n = k + 1$. Now

$$(k + 1)^2 - 7(k + 1) + 12 = k^2 - 5k + 7 = (k^2 - 7k + 12) + (2k - 5) \quad (2)$$

By induction hypothesis $k^2 - 7k + 12 \geq 0$. Also for all $k \geq 3$ we have $(2k - 5) \geq 1 \geq 0$. Thus we have from Equation 2

$$(k + 1)^2 - 7(k + 1) + 12 \geq 0$$

So we prove that for all $n \geq 3$, $(k + 1)^2 - 7(k + 1) + 12 \geq 0$.