

Mid-term Solutions

1 How many ways can you divide n identical chocolates among k boys and l girls such that each boy gets at least 1 chocolate and each girl gets at least 3 chocolates?

Out of the n chocolates, after distributing 1 chocolate to each boy and 3 chocolates to each girl, we are left with $n - k - 3l$ chocolates. The remaining chocolates must be distributed to $k + l$ people. Using balls-and-walls approach, this can be done in $\binom{(n-k-3l)+(k+l)-1}{(n-k-3l)} = \binom{n-2l-1}{n-k-3l}$ ways ■

2 If $T(n) = 3T(n/3) + 1$ and $T(1) = 1$ then guess the expression of $T(n)$. [Assume n is a power of 3.]

Let us assume $n = 3^k$ for some integer k . From the recursive equation in question, we get:

$$\begin{aligned}T(n) &= 3T(n/3) + 1 \\T(n/3) &= 3T(n/9) + 1 \\T(n/9) &= 3T(n/27) + 1\end{aligned}$$

and so on. Using these equations, we get:

$$\begin{aligned}T(n) &= 3T(n/3) + 1 \\&= 3(3T(n/9) + 1) + 1 \\&= 9T(n/9) + 3 + 1 \\&= 9(3T(n/27) + 1) + 3 + 1 \\&= 27T(n/27) + 9 + 3 + 1\end{aligned}$$

and so on and finally we get:

$$\begin{aligned}
T(n) &= 3^k T(n/3^k) + 3^{k-1} + 3^{k-2} + \dots + 9 + 3 + 1 \\
&= 3^k T(1) + 3^{k-1} + 3^{k-2} + \dots + 9 + 3 + 1 \\
&= 3^k + 3^{k-1} + 3^{k-2} + \dots + 3^2 + 3 + 3^0 \\
&= \frac{3^{k+1} - 1}{3 - 1} \\
&= \frac{3 \times 3^k - 1}{2} \\
&= \frac{3n - 1}{2}
\end{aligned}$$

Now, we need to verify the solutions obtained:

$$T(n) = \frac{3n - 1}{2}$$

Note that verification step is essential as this finally proves that our solution is correct.

For base case, $n = 1$, $T(1) = \frac{3-1}{2} = 1$.

Assume that the statement is true for $n/3$, that is $T(n/3) = \frac{3(n/3)-1}{2} = \frac{n-1}{2}$. With this, we get:

$$\begin{aligned}
T(n) &= 3T(n/3) + 1 \\
&= 3 \times \frac{n-1}{2} + 1 \\
&= \frac{3n - 3 + 2}{2} \\
&= \frac{3n - 1}{2}
\end{aligned}$$

Hence, using induction, we can conclude the proof that $T(n) = \frac{3n-1}{2}$. ■

3 If n is even, then give a closed form expression of

$$-1 + 2\binom{n}{1} - 4\binom{n}{2} + 8\binom{n}{3} - 16\binom{n}{4} \dots - \binom{n}{n}2^n$$

Recall Binomial Theorem:

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} (x)^k$$

We can use $x = -2$ in the above theorem to get the sum in the question, but however, the signs in the question are exact opposite of what we'd get by applying binomial theorem. Hence, we need to multiply the whole sum by -1. So, with $x = -2$ and by multiplying by -1, we get:

$$\begin{aligned} -1 + 2\binom{n}{1} - 4\binom{n}{2} + 8\binom{n}{3} - 16\binom{n}{4} \dots - \binom{n}{n}2^n &= -(1-2)^n \\ &= -1 \times (-1)^n \end{aligned}$$

Since n is even, we have $(-1)^n = 1$. So, the final answer is -1. ■

4 If $T(n) = T(n-1) + n$ and $T(1) = 1$, prove that $T(n) = n(n+1)/2$ for all $n \geq 1$

We prove using induction. When $n = 1$, $T(1) = 1(1+1)/2 = 1$. Assume that the statement is true for $n-1$. That is, $T(n-1) = (n-1)n/2$. We shall use this prove the statement for n . We have:

$$\begin{aligned} T(n) &= T(n-1) + n \\ &= \frac{(n-1)n}{2} + n \\ &= n \times \frac{n-1+2}{2} \\ &= n \times \frac{n+1}{2} \\ &= (n+1)n/2 \end{aligned}$$

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5 The lucas sequence 1, 3, 4, 7, 11, 18, 29, ... is defined by $a_1 = 1, a_2 = 3$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. Prove that $a_n < (1.75)^n$.

We shall use induction to prove this. For base case, when $n = 1$, $a_1 = 1 < (1.75)^1$. And, for $n = 2$, $a_2 = 3 < (1.75)^2 = 3.0625$. Here, two base cases are required because the recursive expression for a_n depends on two previous terms, a_{n-1}, a_{n-2} .

For induction hypothesis, assume that the statement is true for $n - 1, n - 2$. That is, $a_{n-1} < (1.75)^{n-1}$ and $a_{n-2} < (1.75)^{n-2}$. So, we have:

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} \\ &< (1.75)^{n-1} + (1.75)^{n-2} \\ &= (2.75) \times (1.75)^{n-2} \\ &< (3.0625) \times (1.75)^{n-2} \\ &= (1.75)^2 \times (1.75)^{n-2} \\ &= (1.75)^n \end{aligned}$$

This proves $a_n < (1.75)^n$. ■

6 How many functions f are there from $\{1, \dots, n\}$ to $\{-1, 0, 1\}$ such that, for every f there exists at least one $i \in \{1, \dots, n\}$ and one $j \in \{1, \dots, n\}$ where $f(i) = 1$ and $f(j) = -1$.

Consider the following sets:

- A = Set of functions where there is at least one $i \in \{1, \dots, n\}$ with $f(i) = 1$
- B = Set of functions where there is at least one $j \in \{1, \dots, n\}$ with $f(j) = -1$

The solution we are looking for is intersection of two sets, $|A \cap B|$ as both the conditions need to be satisfied. We have:

$$|A \cap B| = |A| + |B| - |A \cup B|$$

Let us now compute size of each of these sets.

- $|A| = 3^n - 2^n$. Using complement rule, we count number of functions in which there is no i such that $f(i) = 1$. This means each $f(i)$ has two choices, 0 or -1. So, there are 2^n such functions. Without any conditions, total number of functions is 3^n .

- $|B| = 3^n - 2^n$. Similar argument as previous case.
- $|A \cup B| = 3^n - 1^n = 3^n - 1$. The set $A \cup B$ is set of all functions in which there is at least one $i \in \{1, \dots, n\}$ such that $f(i) = 1$ or one $j \in \{1, \dots, n\}$ such that $f(j) = -1$. Negation of this statement is: There is no i and no j such that $f(i) = 1$ and $f(j) = -1$. This means all elements are mapped to 0. This can be done only in 1 way.

Hence, the answer is:

$$\begin{aligned}
 |A \cap B| &= |A| + |B| - |A \cup B| \\
 &= 3^n - 2^n + 3^n - 2^n - (3^n - 1) \\
 &= 3^n - 2 \times 2^n + 1
 \end{aligned}$$

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