

CSE 20: Assignment Set 6

1. Let $x > -1$ be a real number. Prove that $(1 + x)^n \geq 1 + nx$ for all natural numbers n .

Proof. By induction on n .

Base Case: For $n = 1$, the lefthand side of the inequality is $(1 + x)^1 = 1 + x$ and the righthand side is $1 + 1 \cdot x = 1 + x$. $1 + x \geq 1 + x$, so the inequality is true for the base case.

Inductive Hypothesis: Assume that for some natural number k $(1 + x)^k \geq 1 + kx$ holds.

Inductive Step: We want to show that $(1 + x)^{k+1} \geq 1 + (k + 1)x$ is also true. We start with the inductive hypothesis:

$$\begin{array}{lll} (1 + x)^k & \geq 1 + kx & \text{by inductive hypothesis} \\ (1 + x)^k(1 + x) & \geq (1 + kx)(1 + x) & \text{multiplying } (1 + x) \text{ on both sides} \\ (1 + x)^{k+1} & \geq 1 + x + kx + kx^2 & \\ (1 + x)^{k+1} & \geq 1 + (k + 1)x + kx^2 & \text{factoring } x \text{ on right side} \end{array}$$

We know that k is positive and x^2 must either be 0 or positive because it is a square. Thus, $kx^2 \geq 0$. This means that $1 + (k + 1)x + kx^2 \geq 0$. By transitivity, $(1 + x)^{k+1} \geq 1 + (k + 1)x$, as required. $\square$

2. Prove that $\sum_{i=1}^n i \times i! = (n + 1)! - 1$, for all positive integers n .

Proof. By induction on n .

Base case: For $n = 1$ the lefthand side of the equation is $\sum_{i=1}^1 i \times i! = 1 \times 1! = 1$. The righthand side of the equation is $(1 + 1)! - 1 = 2! - 1 = 1$. Since $1 = 1$, the statement holds for the base case.

Inductive Hypothesis: Assume for some positive integer k , $\sum_{i=1}^k i \times i! = (k + 1)! - 1$.

Inductive Step: We want to show that $\sum_{i=1}^{k+1} i \times i! = (k + 2)! - 1$. We start with the inductive hypothesis:

$$\begin{array}{lll} \sum_{i=1}^k i \times i! & = (k + 1)! - 1 & \text{by inductive hypothesis} \\ \sum_{i=1}^k i \times i! + (k + 1)(k + 1)! & = (k + 1)! - 1 + (k + 1)(k + 1)! & \text{add } (k + 1)(k + 1)! \text{ to both sides} \\ \sum_{i=1}^{k+1} i \times i! & = (k + 1)!(1 + (k + 1)) - 1 & \text{factor } (k + 1)! \text{ on right side} \end{array}$$

By simplifying the right side of the equation we get $\sum_{i=1}^{k+1} i \times i! = (k+2)! - 1$ as required. $\square$

3. Let $\{a_n\}$ be a sequence of natural numbers such that $a_1 = 5$, $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all natural numbers n . Prove that $a_n = 2^n + 3^n$ for all natural number n .

Proof. By strong induction on n .

Base cases: For $n = 1$, $a_1 = 2^1 + 3^1 = 5$, which is true. For $n = 2$, $a_2 = 2^2 + 3^2 = 13$, which is also true.

Inductive Hypothesis: Assume for all natural numbers $m \leq k-1$, $a_m = 2^m + 3^m$

Inductive Step: We want to show that $a_k = 2^k + 3^k$. We start with the recurrence relation:

$$\begin{aligned}
 a_k &= 5a_{k-1} - 6a_{k-2} && \text{from relation definition} \\
 &= 5(2^{k-1} + 3^{k-1}) - 6(2^{k-2} + 3^{k-2}) && \text{by inductive hypothesis} \\
 &= 5 \cdot 2^{k-1} + 5 \cdot 3^{k-1} - 6 \cdot 2^{k-2} - 6 \cdot 3^{k-2} \\
 &= 2^{k-2}(5 \cdot 2 - 6) + 3^{k-2}(5 \cdot 3 - 6) && \text{factoring } 2^{k-2} \text{ and } 3^{k-2} \text{ on right side} \\
 &= 2^{k-2} \cdot 4 + 3^{k-2} \cdot 9 \\
 &= 2^{k-2} \cdot 2^2 + 3^{k-2} \cdot 3^2 \\
 &= 2^k + 3^k && \text{as required.}
 \end{aligned}$$

$\square$

4. Prove that for all $n \geq 1$,

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

Proof. By induction on n .

Base case: For $n = 1$. $1^3 = 1$, $\left(\frac{1(1+1)}{2} \right)^2 = 1$. $1 = 1$ so the statement holds for the base case.

Inductive Hypothesis: Assume for some $k \geq 1$, $1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2} \right)^2$

Inductive Step: We want to show that $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2} \right)^2$. We start with the inductive hypothesis:

$$\begin{aligned}
1^3 + 2^3 + \dots + k^3 &= \left(\frac{k(k+1)}{2}\right)^2 && \text{by inductive hypothesis} \\
1^3 + 2^3 + \dots + k^3 + (k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{add } (k+1)^3 \text{ to both sides} \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2+4(k+1))}{4} && \text{factoring } (k+1)^2 \text{ on numerator} \\
&= \frac{(k+1)^2(k^2+4k+4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4}
\end{aligned}$$

By simplifying the righthand side we get $1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$, as required. $\square$

5. For all $n > 4$ prove that $3^n > 7(2^n)$.

Proof. By induction on n .

Base case: For $n = 5$. $3^5 = 243$ and $7(2^5) = 224$. $243 > 224$, so the statement holds for the base case.

Inductive Hypothesis: Assume for some $k > 4$, $3^k > 7(2^k)$

Inductive Step: We want to show that $3^{k+1} > 7(2^{k+1})$. We start with the inductive hypothesis:

$$\begin{aligned}
3^k &> 7(2^k) && \text{by inductive hypothesis} \\
3^k \cdot 3 &> 7(2^k) \cdot 3 && \text{multiplying 3 on both sides} \\
7(2^k) \cdot 3 &> 7(2^k) \cdot 2 && \text{because } 3 > 2
\end{aligned}$$

So by transitivity and simplification, we get $3^{k+1} > 7(2^{k+1})$, as required. $\square$