

CSE 20: Assignment Set 5

1. If A and B are two sets such that $|A| = 8$ and $|B| = 9$ then can $|A \cap B|$ be equal to 10. Explain your answer.

Answer: No.

The maximum $|A \cap B|$ can be is 8 because there are only 8 elements in A .

2. If A and B are two sets such that $|A| = 8$ and $|B| = 9$ and $|A \cup B| = 15$ then what is $|A \cap B|$.

Answer: 2

By the inclusion-exclusion principle, $|A \cup B| = |A| + |B| - |A \cap B|$.

$$|A \cap B| = |A| + |B| - |A \cup B| = 9 + 8 - 15 = 2$$

3. If $S = \{0, 1\}$ and $T = \{a, e, i, o, u\}$ then what is the set $S \times T$. What is the size of the set $S \times T$.

$$S \times T = \{(0, a), (0, e), (0, i), (0, o), (0, u), (1, a), (1, e), (1, i), (1, o), (1, u)\}$$
$$|S \times T| = |S| \times |T| = 2 \times 5 = 10$$

4. How many functions are there from $\{0, 1\}^n$ to $\{1, 2, \dots, n\}$.

Answer: $n^{(2^n)}$

The number of functions from set X to set Y is $|Y|^{|X|}$ because every element in Y has $|X|$ possible elements it can be mapped from. $|\{0, 1\}| = 2^n$ and $|\{1, 2, \dots, n\}| = n$. So the number of functions from $\{0, 1\}^n$ to $\{1, 2, \dots, n\}$ is $n^{(2^n)}$

5. Prove that for any $n \in \mathbb{Z}^+$, $\sqrt{n} + \sqrt{2}$ is not rational.

Proof. Suppose for sake of contradiction that $\sqrt{n} + \sqrt{2}$ is rational. This means that $\sqrt{n} + \sqrt{2} = \frac{p}{q}$ where p and q and $q \neq 0$.

$$\begin{aligned}
\sqrt{n} + \sqrt{2} &= \frac{p}{q} \\
\sqrt{n} &= \frac{p}{q} - \sqrt{2} && \text{subtracting } \sqrt{2} \text{ from both sides} \\
n &= \frac{p^2}{q^2} - 2\sqrt{2}\frac{p}{q} + 2 && \text{squaring both sides} \\
n &= \frac{p^2 - 2pq\sqrt{2} + 2q^2}{q^2} && \text{rewriting with } q^2 \text{ as denominator} \\
nq^2 &= p^2 - 2pq\sqrt{2} + 2q^2 && \text{multiplying } q^2 \text{ to both sides} \\
nq^2 + 2pq\sqrt{2} &= p^2 + 2q^2 && \text{adding } 2pq\sqrt{2} \text{ to both sides} \\
2pq\sqrt{2} &= p^2 + 2q^2 - nq^2 && \text{subtracting } nq^2 \text{ to both sides} \\
\sqrt{2} &= \frac{p^2 + 2q^2 - nq^2}{2pq} && \text{dividing by } 2pq \text{ on both sides}
\end{aligned}$$

Since p , q , and n are integers, both the numerator and the denominator of the righthand side of the equation must be integers. This means that $\sqrt{2}$ is rational, which contradicts the fact that $\sqrt{2}$ is irrational (as proved in class). $\square$

6. Show that $a^2 + a^4 \equiv 0 \pmod{5}$ if $a \equiv 2 \pmod{5}$ or $a \equiv 3 \pmod{5}$ or if 5 divides a .

Proof. We want to show that if $a \equiv 2 \pmod{5}$ or $a \equiv 3 \pmod{5}$ or if 5 divides a , then $a^2 + a^4 \equiv 0 \pmod{5}$. In other words, we want to show that $a^2 + a^4 = 5k + 0$, where k is an integer. We will do this by considering each of the three cases separately. First let's note some theorems about modular arithmetic that will help us with our proof (Theorem 4, NT-7):

- If $x \equiv m \pmod{d}$ and $y \equiv n \pmod{d}$, then $x + y \equiv m + n \pmod{d}$.
- If $x \equiv m \pmod{d}$, then $x^n \equiv m^n \pmod{d}$.

Case 1: $a \equiv 2 \pmod{5}$

Since $a \equiv 2 \pmod{5}$, then $a^2 \equiv 2^2 \pmod{5}$ and $a^4 \equiv 2^4 \pmod{5}$. $a^2 + a^4 \equiv 2^2 + 2^4 \pmod{5} \equiv 20 \pmod{5}$. By the definition of mod, we can write $a^2 + a^4 = 5b + 20$ where b is an integer, or $a^2 + a^4 = 5(b + 4) + 0$. Since b is an integer, $b + 4$ must also be an integer. Therefore, if $a \equiv 2 \pmod{5}$, then $a^2 + a^4 \equiv 0 \pmod{5}$.

Case 2: $a \equiv 3 \pmod{5}$

Since $a \equiv 3 \pmod{5}$, then $a^2 \equiv 3^2 \pmod{5}$ and $a^4 \equiv 3^4 \pmod{5}$. $a^2 + a^4 \equiv 3^2 + 3^4 \pmod{5} \equiv 90 \pmod{5}$. By the definition of mod, we can write $a^2 + a^4 = 5c + 90$ where c is an integer, or $a^2 + a^4 = 5(c + 18) + 0$. Since c is an integer, $c + 18$ must also be an integer. Therefore, if $a \equiv 3 \pmod{5}$, then $a^2 + a^4 \equiv 0 \pmod{5}$.

Case 3: 5 divides a

By the definition of divides, $a = 5d$, where d is an integer. So, $a^2 + a^4 = (5d)^2 + (5d)^4 = 25d^2 + 625d^4 = 5(5d^2 + 125d^4) + 0$. Since d is an integer, $5d^2 + 125d^4$ must also be an integer. Therefore, if 5 divides a , then $a^2 + a^4 \equiv 0 \pmod{5}$. $\square$

7. Check the truth of the following statements. First convert to logic statements:

- (a) Every even integer is divisible by 4 if and only if either 7 divides 21 or 9 divides 12.
- (b) Either snow is hot or 2 is even implies 3 is even
- (c) $\forall x \in \mathbb{Z}, 3x^2 + 2x + 3 = 0$ implies $\exists x \in \mathbb{Z}$ such that $3x^2 + 2x + 3 = 0$.

(a) **Answer:** False.

Let p = “every even integer is divisible by 4,” which evaluates to false. Let q = “7 divides 21,” which evaluates to true. Let r = “9 divides 12,” which also evaluates to false. Then the statement is $p \iff (q \vee r) \equiv (p \implies (q \vee r)) \wedge ((q \vee r) \implies p)$. We can construct a “truth table” to evaluate the truthness of the statement like so:

p	q	r	$q \vee r$	$p \implies (q \vee r)$	$(q \vee r) \implies p$	$(p \implies (q \vee r)) \wedge ((q \vee r) \implies p)$
F	T	F	T	T	F	F

(b) **Answer:** False.

Let p = “snow is hot,” which evaluates to false. Let q = “2 is even,” which is true. And let r = “3 is even,” which is false. Then the statement is $p \vee (q \implies r)$. Again, we can construct a mini-“truth table.”

p	q	r	$q \implies r$	$p \vee (q \implies r)$
F	T	F	F	F

(c) **Answer:** False.

Let $p = \forall x \in \mathbb{Z}, 3x^2 + 2x + 3 \neq 0$, which is true. Then, the statement $\exists x \in \mathbb{Z}$ such that $3x^2 + 2x + 3 = 0$ is its negation, or $\neg p$, which is false. So the original statement is $p \implies \neg p$.

p	$\neg p$	$p \implies \neg p$
T	F	F

8. Prove that the following expressions are equivalent:

(a) $p \wedge q \implies (p \vee q) \equiv T$

(b) $\neg(p \vee (\neg p \wedge q)) \equiv (\neg p \wedge \neg q)$

(a) Method 1: Truth Table

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \implies (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

The last column shows that $(p \wedge q) \implies (p \vee q) \equiv T$.

Method 2: Algebraic Manipulations

$$\begin{aligned}
 & (p \wedge q) \implies (p \vee q) \\
 & \neg(p \wedge q) \vee (p \vee q) \quad (\text{see Lo-5}) \\
 & (\neg p \vee \neg q) \vee (p \vee q) \quad \text{DeMorgan's rule} \\
 & (\neg p \vee p) \vee (\neg q \vee q) \quad \text{commutativity} \\
 & T \vee T \quad \text{negation rule} \\
 & T \quad \text{bound rule}
 \end{aligned}$$

(b) Method 1: Truth Table

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \vee (\neg p \wedge \neg q)$	$\neg(p \vee (\neg p \wedge \neg q))$	$(\neg p \wedge \neg q)$
T	T	F	F	F	T	F	F
T	F	F	T	F	T	F	F
F	T	T	F	T	T	F	F
F	F	T	T	T	F	T	T

The last two columns are the same, and thus, the statements are equivalent.

Method 2: Algebraic Manipulations

$$\begin{aligned}
 & \neg(p \vee (\neg p \wedge q)) \\
 \neg((p \vee \neg p) \wedge (p \vee q)) & \quad \text{distribution} \\
 \neg(T \wedge (p \vee q)) & \quad \text{negation rule} \\
 \neg(p \vee q) & \quad \text{bound rule} \\
 \neg p \wedge \neg q & \quad \text{DeMorgan's rule}
 \end{aligned}$$

9. Write the following expression using $\neg$, $\vee$ and $\wedge$ only.

(a) $p \implies q$

(b) $p \iff q$

(c) $[\neg p \wedge (p \vee q)] \implies q$

(a) $p \implies q \equiv \neg p \vee q$

(b) $p \iff q \equiv (p \implies q) \wedge (q \implies p) \equiv (\neg p \vee q) \wedge (\neg q \vee p)$

(c) $[\neg p \wedge (p \vee q)] \implies q \equiv \neg[\neg p \wedge (p \vee q)] \vee q$