

CSE 20: Practice Midterm

Maximum marks: 100
Total Time: 50 minutes

1. (20 Marks) Prove that for any $k \in \mathbb{Z}$, if k is an odd integer and not divisible by 3 then $k^2 \equiv 1 \pmod{6}$.

Answer: An integer when divided by 6 can have remainder 0 or 1 or 2 or 3 or 4 or 5.

If the remainder is 0 or 2 or 4 then the integer is $6t$ or $6t + 2$ or $6t + 4$ respectively. In all the cases the integer would be divisible by 2 and hence not odd.

Now if the remainder is 3 in that case the integer is $6t + 3$ which is divisible by 3.

Thus if the integer is odd and not divisible by 3 then the remainder must be 1 or 5. We now do it in a case by case basis

Case 1: (remainder is 1) In this case $k = 6t + 1$ and then $k^2 = 36t^2 + 12t + 1 = 6(6t^2 + 2t) + 1$. Thus $k^2 - 1$ is divisible by 6 and hence $k^2 \equiv 1 \pmod{6}$

Case 2: (remainder is 5) In this case $k = 6t + 5$ and then $k^2 = 36t^2 + 60t + 25 = 6(6t^2 + 12t + 4) + 1$. Thus $k^2 - 1$ is divisible by 6 and hence $k^2 \equiv 1 \pmod{6}$

Since in both the possible cases we have $k^2 \equiv 1 \pmod{6}$ so we have proved it.

2. (20 Marks) Prove that $2^{1/3}$ is not rational.

Answer: Let us prove by contradiction.

So let $2^{1/3}$ is rational. So let $2^{1/3} = p/q$, where both p and q are integers. Since we can assume that the no number divides both p and q , else we can reduce both p and q , so we can assume both p and q cannot be even.

Cubing both sides we have $2 = p^3/q^3$ and so we have

$$2q^3 = p^3 \tag{1}$$

We solve it in case by case basis.

Case 1: (p is odd and q is even) Then in Equation1 the LHS is even while RHS is odd. This cannot be so we have a contradiction.

Case 2: (p is odd and q is odd) In this case also in Equation1 the LHS is even while RHS is odd. This cannot be so we have a contradiction.

Case 3: (p is even and q is odd) Let $p = 2t$. In Equation1 we have $2q^3 = (2t)^3 = 8t^3$ and thus $q^3 = 4t^3$. In this equation thus LHS is odd while RHS is even. This cannot be so we have a contradiction.

Thus in all the possible cases we have a contradiction. So the initial assumption that $2^{1/3}$ is rational is wrong. So $2^{1/3}$ is irrational.

3. (10 each) If A and B are two two sets such that

- (a) If $|A| = 16$ and $|B| = 9$ and A and B does not intersect then what is $|A \cup B|$.
- (b) If $|B| = 9$ what is $|B^3|$.
- (c) If $|A| = 8$ and $|B| = 3$ how many functions are there from $|A|$ to $|B|$.
- (d) If $A \subset B$ and $|A| = 6$ and $|B| = 8$. What is the size of $A^c \cap B$.

Answer: 3a A and B does not intersect. So $|A \cap B| = 0$. Thus $|A \cup B| = |A| + |B| - |A \cap B| = 16 + 9 + 0 = 25$

Answer: 3b $|B^3| = |B|^3 = 9^3$.

Answer: 3c For every element in A the image can be any of the element in B . So the total number of function is $|B|^{|A|} = 3^8$.

Answer: 3d Since A is the subset of B so $|A \cap B| = |A| = 6$. Thus $|A^c \cap B| = |B| - |A \cap B| = 9 - 6 = 3$.

4. (10 Marks) Right the negation of the following statement:

- In all american family there is a member who either likes to play baseball or likes to watch baseball games

Answer: The negation is “There exists an american family where every member is neither play baseball nor watches baseball game”

5. (15 + 15 Marks) Prove that the following of expressions are equivalent:

(a) $p \rightarrow (q \vee r) \equiv (p \wedge \neg q) \rightarrow r \equiv (p \wedge \neg r) \rightarrow q$

(b) $\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q) \equiv p.$

Answer 5a: You have to see that the expressions are equivalent by looking at the truth table.

Answer 5b: By De Morgan's Law

$$\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) = (\neg(\neg p \wedge q) \wedge \neg(\neg p \wedge \neg q)) \quad (2)$$

Now again by De Morgan: $\neg(\neg p \wedge q) = (p \vee \neg q)$ and $\neg(\neg p \wedge \neg q) = (p \vee q)$. Now putting them in Equation 2 we have

$$\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) = (p \vee \neg q) \wedge (p \vee q) \quad (3)$$

Thus we have

$$\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q) = (p \vee \neg q) \wedge (p \vee q) \wedge (p \wedge q) = (p \vee \neg q) \wedge (p \vee q) \quad (4)$$

By distributive law Equation 4 becomes $p \vee (\neg q \wedge q) = p \vee F = p.$

Thus proved.