

CSE 20

Lecture 9: Propositional Logic (contd...)

Proof Techniques

To prove statement B from A :

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- Direct Proof:

$$A \implies B$$

Proof Techniques

To prove statement B from A :

- Direct Proof:

$$A \implies B$$

- Proof by contradiction:

$(\neg B \wedge A)$ gives a contradiction

Some Number Theory

If a, b are two positive integers then b divides a if $a = bq$ for some positive integer q .

Some Number Theory Problems

If a, b, p are three positive integers such that a and b are divisible by p then prove that p divides $a + b$.

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- p divides a implies $a = pr$, for some positive integer r .
- Similarly p divides b implies $b = ps$, for some positive integer s .
- So $a + b = pr + ps = p(r + s)$.
- Since $r + s$ is a positive integer so p divides $a + b$.

What is a remainder?

Let a, d be two positive integers.

If a can be written as $dq + r$ where q and r are positive integers and $r < d$ then r is the remainder when a is divided by d .

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Let a, d be two positive integers.

If a can be written as $dq + r$ where q and r are positive integers and $r < d$ then r is the remainder when a is divided by d .

In other words, if d divided $a - r$ when $r < a$ then r is the remainder when a is divisible by d

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If a, b are two positive integers then b divides a if $a = bq$ for some positive integer q .

If a, b are two positive integers then b does not divide a if $a = bq + r$ for some positive integer q and r , and $1 \leq r < b$

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- So $a + b = pr + ps + t = p(r + s) + t$.

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If a, b, p are three positive integers such that a is divisible by p and b is not divisible by p then prove that p does not divide $a + b$.

- p divides a implies $a = pr$, for some positive integer r .
- Similarly p does not divide b implies $b = ps + t$, for some positive integer s, t and $1 \leq t < p$.
- So $a + b = pr + ps + t = p(r + s) + t$.
- Since $r + s$ is a positive integer so p divides $(a + b) - t$.
- Since $1 \leq t < p$ so p does not divide $(a + b)$

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- So $ab = pr.qs = pq(rs)$

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- So $ab = pr.qs = pq(rs)$
- So pq divides ab

Question from the Assignment 3

If a and b are two positive integers then prove that $a^2 - 4b$ cannot be equal to 2.

iclicker question

If a positive integer a is divided by 4 then what are the possible remainders

- 1, 2, 3, 4
- 0, 1, 2, 3,
- 0, 1, 2, 3, 4
- Any positive integer is possible

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We will solve in in case by case basis.

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We split the problem into 4 case depending on the remainder when a is divided by 4.

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Case 1 The remainder when a is divided by 4 is 0

Case 2 The remainder when a is divided by 4 is 1

Case 3 The remainder when a is divided by 4 is 2

Case 4 The remainder when a is divided by 4 is 3

Question from the Assignment 3: Case 1

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If a and b are two positive integers then prove that $a^2 - 4b$ cannot be equal to 2.

Case 1 The remainder when a is divided by 4 is 0

- $a = 4r$ for some positive integer r .

Question from the Assignment 3: Case 1

If a and b are two positive integers then prove that $a^2 - 4b$ cannot be equal to 2.

Case 1 The remainder when a is divided by 4 is 0

- $a = 4r$ for some positive integer r .
- So $a^2 = 16r^2$.

Question from the Assignment 3: Case 1

If a and b are two positive integers then prove that $a^2 - 4b$ cannot be equal to 2.

Case 1 The remainder when a is divided by 4 is 0

- $a = 4r$ for some positive integer r .
- So $a^2 = 16r^2$.
- Thus $a^2 - 4b = 16r^2 - 4b = 4(4r^2 - b)$

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- $a = 4r$ for some positive integer r .
- So $a^2 = 16r^2$.
- Thus $a^2 - 4b = 16r^2 - 4b = 4(4r^2 - b)$
- Since $4r^2 - b$ is an integer and 4 times an integer can never be 2 so $a^2 - 4b$ cannot be equal to 2.

Question from the Assignment 3: Case 2

If a and b are two positive integers then prove that $a^2 - 4b$ cannot be equal to 2.

Case 2 The remainder when a is divided by 4 is 1

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If a and b are two positive integers then prove that $a^2 - 4b$ cannot be equal to 2.

Case 2 The remainder when a is divided by 4 is 1

- $a = 4r + 1$ for some positive integer r .

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If a and b are two positive integers then prove that $a^2 - 4b$ cannot be equal to 2.

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- $a = 4r + 1$ for some positive integer r .
- So $a^2 = 16r^2 + 8r + 1$.

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If a and b are two positive integers then prove that $a^2 - 4b$ cannot be equal to 2.

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- So $a^2 = 16r^2 + 8r + 1$.
- Thus $a^2 - 4b = 16r^2 + 8r + 1 - 4b = 4(4r^2 + 2r - b) + 1$

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- Thus $a^2 - 4b = 16r^2 + 8r + 1 - 4b = 4(4r^2 + 2r - b) + 1$
- Since $4r^2 + 2r - b$ is an integer and 4 times an integer can never be 1
- so $4(4r^2 + 2r - b) + 1$ cannot be equal to 2
- and so $a^2 - 4b$ cannot be equal to 2.

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If a and b are two positive integers then prove that $a^2 - 4b$ cannot be equal to 2.

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- $a = 4r + 2$ for some positive integer r .
- So $a^2 = 16r^2 + 16r + 4$.

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- Thus $a^2 - 4b = 16r^2 + 16r + 4 - 4b = 4(4r^2 + 4r + 1 - b)$

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If a and b are two positive integers then prove that $a^2 - 4b$ cannot be equal to 3.

Case 4 The remainder when a is divided by 4 is 3

- $a = 4r + 3$ for some positive integer r .
- So $a^2 = 16r^2 + 24r + 9$.

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$$a^2 - 4b = 16r^2 + 24r + 9 - 4b = 4(4r^2 + 6r + 2 - b) + 1$$

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- so $4(4r^2 + 6r + 2 - b) + 1$ cannot be equal to 2
- and so $a^2 - 4b$ cannot be equal to 2.

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If a positive integer a is divided by 4 then the possible remainders are 0, 1, 2 and 3.

We will solve in in case by case basis.

We split the problem into 4 case depending on the remainder when a is divided by 4 and show that for every case $a^2 - 4b$ cannot be equal to 2.

Notation

If r is the remainder when a is divided by d it is represented as

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If a and b are two positive integers then prove that $a^2 - 4b$ cannot be equal to 2,]colored IMPLIES

Thus we have proved that for any positive integer a

$$a^2 \equiv 0(\text{mod } 4), \text{ OR } a^2 \equiv 1(\text{mod } 4), \text{ or}$$

Proof of the problem using contradiction

To prove: If a and b are two positive integers then prove that $a^2 - 4b$ cannot be equal to 2,

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- ❶ If a and b are not positive integers then $a^2 - 4b$ can be equal to 2,
- ❷ If $a^2 - 4b$ cannot be equal to 2 and a and b are positive integers then we get a contradiction
- ❸ If $a^2 - 4b$ cannot be equal to 2 and a and b are positive integers then there is no contradiction.
- ❹ If $a^2 - 4b$ can be equal to 2 and a and b are positive integers then we get a contradiction

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If $a^2 - 4b$ can be equal to 2 and a and b are positive integers then we get a contradiction

- $a^2 - 4b = 2 \implies a^2 = 4b + 2 \implies a^2 - 2 = 4b$

Proof of the problem using contradiction

If $a^2 - 4b$ can be equal to 2 and a and b are positive integers then we get a contradiction

- $a^2 - 4b = 2 \implies a^2 = 4b + 2 \implies a^2 - 2 = 4b$
- Thus 4 divides $a^2 - 2$

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If $a^2 - 4b$ can be equal to 2 and a and b are positive integers then we get a contradiction

- $a^2 - 4b = 2 \implies a^2 = 4b + 2 \implies a^2 - 2 = 4b$
- Thus 4 divides $a^2 - 2$
 $= (a^2 - 1) - 1 = (a - 1)(a + 1) - 1.$
- If a is even then 4 divides a^2 and so a contradiction.

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- $a^2 - 4b = 2 \implies a^2 = 4b + 2 \implies a^2 - 2 = 4b$
- Thus 4 divides $a^2 - 2$
 $= (a^2 - 1) - 1 = (a - 1)(a + 1) - 1$.
- If a is even then 4 divides a^2 and so a contradiction.
- If a is odd then 2 divides $(a - 1)$ and 2 divides $(a + 1)$ so 4 divides $(a - 1)(a + 1) = a^2 - 1$ and so a contradiction.

Direct Proof or Proof by Contradiction

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But in some cases one of the techniques is easier than the other.

Is $\sqrt{2}$ a rational?

Prove that $\sqrt{2}$ is not rational.

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To prove by contradiction what do have to prove:

- 2 is a rational number
- $\sqrt{2}$ cannot be written as $\frac{p}{q}$ for any positive integer p and q .
- If $\sqrt{2} = \frac{p}{q}$ for some positive integers p and q then there is some problem
- $\sqrt{2}$ is not an integer.

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