

# CSE 20

## Lecture 8: Propositional Logic (contd...)

# Outline of the course

## Quizzes and Midterm:

- Quiz 2: on 3rd February
- Midterm: on 14th February
- Quiz 3: on 26th February
- Quiz 4: on 5th March
- Quiz 5: on 12th March

## Question from Quiz

Part 3a: Show that if  $a$  and  $b$  are integers in the range 1 through 256, and the sum of  $a$  and  $b$  is also in this range, then

$$2^9 \leq (2^9 - a) + (2^9 - b) < 2^{10}.$$

Part 3b: *Note:*  $2^9 = 512$ . Explain why it follows that the binary representation of  $(2^9 - a) + (2^9 - b)$  has a leading term in the  $2^9$ th position.

# Rephrasing the question

Show that if  $a$  and  $b$  are integers in the range 1 through 256, and the sum of  $a$  and  $b$  is also in this range, then

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**IMPLIES**

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**IMPLIES**  $2^9 \leq (2^9 - a) + (2^9 - b)$ , **AND**

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- $b$  is an integers in the range 1 through 256, **AND**
- the sum of  $a$  and  $b$  is in the range 1 through 256,

$$\text{IMPLIES } 2^9 \leq (2^9 - a) + (2^9 - b), \text{ AND} \\ (2^9 - a) + (2^9 - b) < 2^{10}.$$

- $a$  is an integer in the range 1 through 256, ( $=p$ ) AND
- $b$  is an integer in the range 1 through 256, ( $=q$ ) AND
- the sum of  $a$  and  $b$  is in the range 1 through 256, ( $=r$ )

IMPLIES

$$2^9 \leq (2^9 - a) + (2^9 - b), \text{ ( $=s$ ) AND} \\ (2^9 - a) + (2^9 - b) < 2^{10}. \text{ ( $=t$ )}$$

To Prove:

$$(p \wedge q \wedge r) \implies (s \wedge t)$$

# How to prove the problem

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Notice that:

$$(r \implies s) \wedge (r \implies t)$$

IMPLIES

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# How to prove the problem

To Prove:  $(p \wedge q \wedge r) \implies (s \wedge t)$

Notice that:

$$(r \implies s) \wedge (r \implies t)$$

IMPLIES

$$(p \wedge q \wedge r) \implies (s \wedge t)$$

Similarly,

$$(p \wedge q) \implies (s \wedge t) \text{ IMPLIES } (p \wedge q \wedge r) \implies (s \wedge t).$$

# Rephrasing in Propositional Logical

Show that if

- $a$  is an integers in the range 1 through 256, ( $=p$ ) AND
- $b$  is an integers in the range 1 through 256, ( $=q$ ) AND
- the sum of  $a$  and  $b$  is in the range 1 through 256, ( $=r$ )

IMPLIES

$$2^9 \leq (2^9 - a) + (2^9 - b), \text{ ( $=s$ ) AND} \\ (2^9 - a) + (2^9 - b) < 2^{10}. \text{ ( $=t$ )}$$

To Prove:

$$(r \implies s) \wedge (r \implies t)$$

# Direct proof of the statement

Show that if

- the sum of  $a$  and  $b$  is in the range 1 through 256, (=r)

IMPLIES

$$2^9 \leq (2^9 - a) + (2^9 - b), \text{ (=s) AND} \\ (2^9 - a) + (2^9 - b) < 2^{10}. (=t)$$

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To Prove:  $(r \implies s) \wedge (r \implies t)$

$$(2^9 - a) + (2^9 - b) = 2^{10} - (a + b) < 2^{10} \text{ [Since } (a + b) \geq 1]$$

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To Prove:  $(r \implies s) \wedge (r \implies t)$

$$(2^9 - a) + (2^9 - b) = 2^{10} - (a + b) < 2^{10} \text{ [Since } (a + b) \geq 1]$$

$$(2^9 - a) + (2^9 - b) = 2^9 + (2^9 - (a + b)) \geq 2^9$$

$$\text{[Since } 2^9 \geq (a + b)]$$

# Proof Techniques

To prove statement  $B$  from  $A$ :

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To prove statement  $B$  from  $A$ :

- Direct Proof:

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- Proof by contradiction:

$(\neg B \wedge A)$  gives a contradiction

# Proof of the statement by contradiction

Show that if

- the sum of  $a$  and  $b$  is in the range 1 through 256, (=r)

IMPLIES

$$2^9 \leq (2^9 - a) + (2^9 - b), \quad (=s)$$

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To Prove:  $(\neg s \wedge r)$  gives a contradiction

For proving the above statement using contradiction what do we have to prove?

# Proof of the statement by contradiction

Show that if the sum of  $a$  and  $b$  is in the range 1 through 256 THEN  $2^9 \leq (2^9 - a) + (2^9 - b)$ .

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- 1 If  $2^9 \leq (2^9 - a) + (2^9 - b)$  then sum of  $a$  and  $b$  is in the range 1 through 256
- 2 If  $2^9 < (2^9 - a) + (2^9 - b)$  then sum of  $a$  and  $b$  is in the range 1 through 256
- 3 If  $2^9 > (2^9 - a) + (2^9 - b)$  AND the sum of  $a$  and  $b$  is in the range 1 through 256 then there is A PROBLEM
- 4 If  $2^9 \geq (2^9 - a) + (2^9 - b)$  AND the sum of  $a$  and  $b$  is in the range 1 through 256 then there is NO PROBLEM

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$$2^9 > (2^9 - a) + (2^9 - b) = 2^{10} - (a + b)$$

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$$\implies (a + b) > 2^{10} - 2^9$$

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$$\implies 2^9 = 512 < (a + b)$$

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$$\implies 2^9 = 512 < (a + b) \leq 256 \text{ [From } r\text{].}$$

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$$\implies (a + b) > 2^{10} - 2^9 = 2^9$$

$$\implies 2^9 = 512 < (a + b) \leq 256 \text{ [From } r\text{].}$$

Thus,  $512 < 256$  **CONTRADICTION**

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- Contrapositive
- Induction

Depending on what the problem is we will have to decide which proof technique to use.

# Why is proof by contradiction correct?

Why is proving  $A \implies B$  same as proving

$(\neg B \wedge A)$  gives a contradiction

In other words

$A \implies B$  is same as

$$(\neg B \wedge A) = \text{FALSE}$$

# Checking Equivalence

## Definition

Two statements are equivalent if their TRUTHTABLES are the same.

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| $A$ | $B$ | $A \implies B$ | $(\neg B \wedge A) = F$ |
|-----|-----|----------------|-------------------------|
| F   | F   |                |                         |
| F   | T   |                |                         |
| T   | F   |                |                         |
| T   | T   |                |                         |

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|-----|-----|----------------|-------------------------|
| F   | F   | T              |                         |
| F   | T   | T              |                         |
| T   | F   | F              |                         |
| T   | T   | T              |                         |

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|-----|-----|----------------|-------------------------|
| F   | F   | T              | T                       |
| F   | T   | T              | T                       |
| T   | F   | F              | F                       |
| T   | T   | T              | T                       |

# Proof of “Proof by Contradiction”

Since  $A \implies B$  is equivalent to  $(\neg B \wedge A) = \text{FALSE}$  so for proving  $A \implies B$  its enough to show  $(\neg B \wedge A) = \text{FALSE}$ .

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Similarly many other statements can be shown to be equivalent

Sometimes they can be called RULES.

# Rules about connective

For example:  $(A \implies B)$  can be replaced to  $(\neg B \wedge \neg A)$

In other words:  $(A \implies B) \iff (\neg A \vee B)$

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How to prove this rule?

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How to prove this rule?

TRUTHTABLE

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| $A$ | $B$ | $A \implies B$ | $\neg A \vee B$ |
|-----|-----|----------------|-----------------|
| F   | F   |                |                 |
| F   | T   |                |                 |
| T   | F   |                |                 |
| T   | T   |                |                 |

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To prove:  $(A \implies B) \iff (\neg A \vee B)$

| $A$ | $B$ | $A \implies B$ | $\neg A \vee B$ |
|-----|-----|----------------|-----------------|
| F   | F   | T              |                 |
| F   | T   | T              |                 |
| T   | F   | F              |                 |
| T   | T   | T              |                 |

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To prove:  $(A \implies B) \iff (\neg A \vee B)$

| $A$ | $B$ | $A \implies B$ | $\neg A \vee B$ |
|-----|-----|----------------|-----------------|
| F   | F   | T              | T               |
| F   | T   | T              | T               |
| T   | F   | F              | F               |
| T   | T   | T              | T               |

# Equivalence of statements/propositions

If the truth tables of two statement/propositions are identical then the two statement/propositions are equivalent

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If the truthtables of two statement/propositions are identical then the two statement/propositions are equivalent

Thus:  $A \implies B$  is equivalent to  $\neg A \vee B$ .

Similarly:

- $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$  (Distributive rule)
- $\neg(p \wedge q) \iff (\neg p \vee \neg q)$  (De Morgan's rule)
- More rules in Theorem 1 in Section 1 of Unit:Lo from Bender Williamson book

## So for statements/propositions

- A statement is correct iff it is true for all inputs
- Checking a statement is correct is done using TRUTHTABLE.
- Two statements are equivalent if their TRUTHTABLES are identical.
- A statement which is always true is called a TAUTOLOGY.
- A statement that is never true is called a CONTRADICTION.