Name: ______________________

Student ID:____________________

CSE 21A

Midterm #2

February 28, 2013

There are 6 problems. The number of points a problem is worth is shown next to the problem. Show your work (even on multiple choice questions)! Also, make sure you write legibly so that I have a chance of being able to read your solutions! Additional scratch paper is available at the front of the room. This is a CLOSED BOOK test. However, you may use one 8 1/2 by 11 inch sheets of paper with hand-written notes (on both sides). You can use a calculator if you wish but it shouldn't be necessary since answers can be left in unexpanded form, i.e., using ! and $\binom{x}{y}$.Good luck!

Prob.	Score
1	
2	
3	
4	
5	
6	
Total	

1. [20 points] An urn contains 4 Red and 2 Blue marbles. A fair coin is flipped. If the flip comes up Heads, one Red marble is added to the urn and one Blue marble is removed from the urn. One the other hand, if the flip comes up Tails, then one Red marble is removed from the urn and one Blue marble is added to the urn. Now a random marble M is drawn from the urn.

(i) What is the probability that M is Red?

Solution

$Pr(M = R) = Pr(M = R \,|\, T) \cdot Pr(T) + Pr(M = R \,|\, H) \cdot Pr(H) = \frac{3}{6} \cdot \frac{1}{2} + \frac{5}{6} \cdot \frac{1}{2} = \frac{2}{3}$.

(ii) What is the probability that the flip is Heads, given that M is Blue?

Solution

$Pr(H \,|\, M = B) = Pr(H \cap M = B)/Pr(M = B) = (\frac{1}{2} \cdot \frac{1}{6})/(\frac{1}{3}) = \frac{1}{4}$.

2. [10 points] A biased coin has $Pr(H) = \alpha$ and $Pr(T) = 1 - \alpha$ (where H denotes Heads and T denotes Tails). The coin is flipped n times. What is the expected number of occurrences of the consecutive pattern THH? (For example, in the sequence $HHTTHHTHTHHHTT$ there are 2 occurrences of THH.)

(i) $n\alpha(1-\alpha)^2$;

(ii) $n\alpha^2(1-\alpha)$;

(iii) $(n-1)\alpha(1-\alpha)^2$;

(iv) $(n-2)\alpha^2(1-\alpha)$;

(v) none of the above.

Answer

(iv) (For more detail, consider the Solution to Problem 5.6 on Homework 5.)

3. [20 points] A random hand H of 5 cards is drawn from an ordinary deck of 52. What is the probability that H has at least two Spades, given that H has the Ace of Spades?

Solution

Let's find the answer by considering the complement, i.e. that H has at most one Heart:
$Pr(\geq 2S \mid AS) = 1 - Pr(\leq 1S \mid AS) = 1 - \frac{Pr(\leq 1S \cap AS)}{Pr(AS)} = 1 - \frac{\binom{39}{4}/\binom{52}{5}}{\binom{51}{4}/\binom{52}{5}} = 1 - \frac{\binom{39}{4}}{\binom{51}{4}}$.
We get the $\binom{39}{4}$ factor because once we've picked the Ace of Spades, the remaining 4 cards on the hand will be from the other 3 suits.
(Note that this is almost the same problem as Problem 5.1 (iii) on Homework 5.)

4. [15 points] The Acme Parachute Company produces (rather questionable) parachutes. It is known that with probability α, a randomly selected parachute will be defective (so with probability $1-\alpha$, the parachute is good). However, there is a test T which behaves as follows. If T is applied to a good parachute, then with probability β, the test will indicate that the parachute is good (and with probability $1-\beta$, T will say that it is defective). Also, if T is applied to a defective parachute, then with probability γ, it will say that the parachute is good (and with probability $1-\gamma$, it will indicate defective).

What is the probability that a random parachute P is good given that the test T says it is defective? In other words, what is $Pr(P \text{ is good} \,|\, T \text{ says defective})$?

(i) $\frac{(1-\alpha)(1-\gamma)}{(1-\alpha)(1-\gamma)+\alpha(1-\beta)}$;

(ii) $\frac{(1-\alpha)(1-\beta)}{(1-\alpha)(1-\gamma)+\alpha(1-\beta)}$;

(iii) $\frac{(1-\alpha)(1-\beta)}{(1-\alpha)(1-\beta)+\alpha(1-\gamma)}$;

(iv) $\frac{\alpha(1-\gamma)}{(1-\alpha)(1-\gamma)+\alpha(1-\beta)}$;

(v) none of the above.

Answer

(iii)
(You can use Bayes' Rule.)

5. [15 points] A fair coin is flipped 3 times resulting in the flip sequence $F_1F_2F_3$. Consider the three events:

E_1 : F_1 is Heads;

E_2 : F_1 and F_2 are *different*;

E_3: an *odd* number of the F_i are Tails.

Which of the *pairs* of these events are *independent*?

(i) Only E_1 and E_2 are independent;

(ii) Only E_1 and E_3 are independent;

(iii) All three pairs are independent;

(iv) Only the pairs E_1, E_2 and E_2, E_3 are independent;

(v) None of the above answers is correct.

Solution

We see that each of these events has 4 elements: $E_1 = \{HHH, HHT, HTH, HTT\}$, $E_2 = \{HTH, HTT, THH, THT\}$, $E_3 = \{HHT, HTH, THH, TTT\}$, thus the same probability. $Pr(E_1) = Pr(E_2) = Pr(E_3) = \frac{4}{8} = \frac{1}{2}$. So, any two of these events are independent if and only if their intersection has probability $\frac{1}{4}$, or in other words contains 2 elements. It is easy to verify that this is the case with all pairs, so the answer is (iii).

6. [20 points] An urn contains 2 Red, 3 White and 4 Blue balls.

(i) Three balls are drawn at random *without replacement*. What is the probability that at *least two* of the balls have the same color?

Solution

Let's consider the complement, where each ball drawn has a different color, i.e. one of each color: $1 - \frac{\binom{2}{1}\binom{3}{1}\binom{4}{1}}{\binom{9}{3}} = 1 - \frac{2}{7} = \frac{5}{7}$. (See also the solution to Problem 4.6 on Homework 4.)

(ii) Three balls are drawn randomly one at a time *with replacement*. What is the probability that *at least two* of the balls have the same color?

Solution

Let's consider the complement again: $1 - 3! \cdot (\frac{2}{9})(\frac{3}{9})(\frac{4}{9}) = 1 - \frac{16}{81} = \frac{65}{81}$. (See solution to Problem 4.6 (ii) on Homework 4.)

Scratch paper