Name: _____

Student ID:_____

CSE 21B

Midterm #1

January 31, 2013

There are 7 problems. The number of points a problem is worth is shown next to the problem. Show your work. Also, make sure you write legibly so that I have a chance of being able to read your solutions! Additional scratch paper is available at the front of the room. This is a CLOSED BOOK test. However, you may use one 8 1/2 by 11 inch sheets of paper with hand-written notes (on both sides). You can use a calculator if you wish but it shouldn't be necessary since answers can be left in unexpanded form, i.e., using ! and $\binom{x}{y}$.Good luck!

Prob.	Score
1	
2	
3	
4	
5	
6	
7	
Total	

1. [10 points] How many 5-card hands can be formed from an ordinary deck of 52 cards if *not* all 4 suits occur in the hand (i.e., at least one suit is missing from the hand)?

$$\binom{52}{5} - 4\binom{13}{2}\binom{13}{1}^3$$

To count the ways that don't work: 4 ways to choose a suit to use twice, $\binom{13}{2}$ ways to choose two cards from that suit, $\binom{13}{1}$ ways to choose one card from the remaining suits. There are $\binom{52}{5}$ ways to count total.

2. [10 points] A shelf contains 20 books. How many ways can 5 books be selected from these 20 with the restriction that no 2 selected books can be adjacent?

(i) $\binom{20}{5}$;

(ii) $\binom{16}{5}$;

(iii) $\binom{20}{4}$;

(iv) $\binom{16}{4}$;

(v) none of the above.

$$\binom{16}{5}$$

5 books means we kill 4 slots leaving 16 slots remaining for 5 books

3. [20 points] How many ways can 12 identical jellybeans be distributed to 2 boys and 4 girls if every girl gets **at least one** jellybean and every boy gets **at most one** jellybean?

$$\binom{11}{3} + 2\binom{10}{3} + \binom{9}{3}$$

One way to give zero jellybeans to boys then $\binom{11}{3}$ ways to give jellybeans to girls (3 bars, $12 - 0 - 4$ stars)
Two ways to give one jellybean to boys then $\binom{10}{3}$ ways to give jellybeans to girls (3 bars, $12 - 1 - 4$ stars)
One way to give two jellybeans to boys then $\binom{9}{3}$ ways to give jellybeans to girls (3 bars, $12 - 2 - 4$ stars)

4. [10 points] How many different arrangements are there using **all** the letters of **KNICKKNACK**?

$$\frac{10!}{2!2!4!}$$

By the bookkeeper rule

5. (a) [5 points] What is the coefficient of x^5y^7 in the expansion of $(3x-4y)^{12}$?

$$3^5(-4)^7\binom{12}{5}$$

3^5 from x, $(-4)^7$ from y and $\binom{12}{5}$ counting ways to distribute the exponents accordingly.

(b) [5 points] What is the coefficient of $a^4b^4c^2$ in the expansion of $(a+b+2c)^{10}$?

$$2^2\frac{10!}{2!4!4!}$$

2^2 from c then the rest from the multinomial coefficient

6. We want to count step-by-step paths between points with integer coordinates. Only two kinds of steps are allowed: a right-step which increments the x coordinate and an up-step which increments the y coordinate.

(a) [10 points] How many paths are there from point $(0, 0)$ to point $(12, 12)$?

$$\binom{24}{12}$$

See homework 2

(b) [10 points] How many paths are there when there is an impassable boulder sitting at point $(5, 6)$?

$$\binom{24}{12} - \binom{11}{5}\binom{13}{7}$$

See homework 2

7. [10 points] The generating function for the sequence $\langle 1, 1, 1, 1, \ldots \rangle$ is $1 + x + x^2 + x^3 \ldots = \frac{1}{1-x}$. What is a formula for the generating function for the sequence $\langle 1, 1, 1, 2, 2, 2, 3, 3, 3, \ldots \rangle$?

(a) $\frac{1+x}{(1-x)^3}$

(b) $\frac{1+x+x^2}{(1-x^2)^3}$

(c) $\frac{1}{(1-x^3)^2}$

(d) $\frac{1+x+x^2}{(1-x^3)^2}$

(e) None of the above.

(d)

$1 + 2x + 3x^2 + 4x^3 \ldots = \frac{1}{(1-x)^2}$
so from taking the derivative
$1 + 2x^3 + 3x^6 + 4x^9 + \ldots = \frac{1}{(1-x^3)^2}$,
$x + 2x^4 + 3x^7 + 4x^{10} + \ldots = \frac{x}{(1-x^3)^2}$
$x^2 + 2x^5 + 3x^8 + 4x^{11} + \ldots = \frac{x^2}{(1-x^3)^2}$

Adding these we get $1 + x + x^2 + 2x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 3x^8 + \ldots = \frac{1+x+x^2}{(1-x^2)^3}$.

An alternative solution used by a number of students is the following.
From $\langle 1, 1, 1, 1, 1, 1, \ldots \rangle = \frac{1}{1-x}$, we repeatedly shift by 3 places and add:

$\langle 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots \rangle + \langle 0, 0, 0, 1, 1, 1, 1, 1, 1, \ldots \rangle + \langle 0, 0, 0, 0, 0, 0, 1, 1, 1, \ldots \rangle =$
$\frac{1}{1-x} + \frac{x^3}{1-x} + \frac{x^6}{1-x} + \cdots = \frac{1+x^3+x^6=\ldots}{1-x} = \frac{1}{(1-x)(1-x^3)} = \frac{1+x+x^2}{(1-x^2)^3}$.

Scratch paper