CSE21 WI13

Homework #5

[Each problem is worth 20 points. This set is challenging!]

5.1 A hand H of 3 random cards are dealt from an ordinary deck of 52. Let E_1 denote the event that H has at least 1 Ace, E_2 denote the event that H has at least 2 Aces, and let E_{AS} denote the event that H includes the Ace of Spades.
(i) What are $Pr(E_1), Pr(E_2)$ and $Pr(E_{AS})$?
(ii) What is the conditional probability $Pr(E_2 \,|\, E_1)$?
(iii) What is the conditional probability $Pr(E_2 \,|\, E_{AS})$?
(Are you surprised that the answers to (ii) and (iii) are different?)

Solution

(i) $Pr(E_1) = 1 - \frac{\binom{48}{3}}{\binom{52}{3}},\ Pr(E_2) = \frac{\binom{4}{3}+\binom{4}{2}\binom{48}{1}}{\binom{52}{3}},\ Pr(E_{AS}) = \frac{\binom{51}{2}}{\binom{52}{3}};$

(ii) $Pr(E_2 \,|\, E_1) = \frac{Pr(E_2 \cap E_1)}{Pr(E_1)} = \frac{Pr(E_2)}{Pr(E_1)} = \frac{\binom{4}{3}+\binom{4}{2}\binom{48}{1}}{\binom{52}{3}-\binom{48}{3}};$

(iii) $Pr(E_2 \,|\, E_{AS}) = 1 - \frac{\binom{48}{2}}{\binom{51}{2}}.$

5.2 An urn contains 3 Red and 4 White marbles. A fair coin is flipped. If the flip is Heads then 1 Red and 2 White marbles are added to the urn. On the other hand, if the flip is Tails, then 1 Red and 2 White marbles are *removed* from the urn. Two random marbles are now drawn from the urn without replacement.
(i) What is the probability that both of the drawn marbles are White?
(ii) What is the probability that the flip was Heads, given that the two drawn marbles have different colors?

Solution

(i) $Pr(2W) = Pr(2W \,|\, H)\, P(H) + Pr(2W \,|\, T)\, P(T) = \frac{1}{2}\left(\frac{\binom{6}{2}}{\binom{10}{2}} + \frac{\binom{2}{2}}{\binom{4}{2}}\right) = \frac{1}{4}.$

(ii) $Pr(\text{one } R, \text{one } W) = \frac{1}{2}\frac{2!\binom{4}{1}\binom{6}{1}}{\binom{10}{2}} + \frac{1}{2}\frac{2!\binom{2}{1}\binom{2}{1}}{\binom{4}{2}} = \frac{24}{45} + \frac{4}{6} = \frac{3}{5}$, so we get
$Pr(H \,|\, \text{one } R, \text{one } W) = \frac{Pr(H \cap (\text{one } R, \text{one } W))}{Pr(\text{one } R, \text{one } W)} = (\frac{1}{2}\frac{24}{45})/\frac{3}{5} = \frac{4}{9}.$

5.3 Two teams A and B compete in a “best-of-5” competition. This means they play each other until one team has won 3 games. Suppose that for any of the games, the probability that A beats B is α. What is the probability that A wins the "best-of-5" competition?

Solution

Assuming A wins the competition, the last game will be won by A. We now split the problem into cases according to how many games it takes for A to win the competition:

. 3 games all of A wins

. 4 games, implying A wins 2 of the first 3.

. 5 games, implying A wins only 2 of the first 4.

When we sum this up, we get
$Pr(\text{AAA})+\binom{3}{1}Pr(\text{AABA})+\binom{4}{2}Pr(\text{AABBA}) = \alpha^3+3\alpha^3(1-\alpha)+6\alpha^3(1-\alpha)^2).$

5.4 A fair coin if flipped 3 times. If (F_1, F_2, F_3) denotes a typical flip sequence, let E_1 denote the event that *at least two* of the $F_i's$ are Heads, let E_2 denote the event that *exactly two* of the $F_i's$ are Heads, and let E_3 denote the event that all the F_i are the same. Which of the pairs of these three events are independent?

Solution

The only pair that is independent is E_1 and E_3.
Indeed,since $Pr(E_1) = (\frac{1}{2})^3 + \binom{3}{2}(\frac{1}{2})^3 = \frac{1}{2}$ and $Pr(E_3) = 2 \cdot (\frac{1}{2})^3 = \frac{1}{4}$, so $Pr(E_1) \cap E_3) = Pr(\text{all Heads}) = \frac{1}{8} = Pr(E_1)Pr(E_3)$, while $Pr(E_1) \cap E_2) = Pr(E_2)$ and $Pr(E_2) \cap E_3) = 0$.

5.5 Two random cards are drawn one at a time without replacement from a deck of 52.
(i) What is the probability that the second card is an Ace?
(ii) What is the probability that the second card is an Ace, given that the first card drawn was a King?
(iii) What is the probability that the second card is an Ace, given that the first card drawn was an Ace?

Solution

(i) $\frac{4}{52}$;
(ii) $\frac{4}{51}$;
(iii) $\frac{3}{51}$

5.6 A biased coin C has $Pr(Heads) = \alpha$ and $Pr(Tails) = 1 - \alpha$. The coin is flipped n times.
What is the expected number of Heads that will occur?
(*Optional*) What is the expected number of times that the sequence HT will occur? (For example, in the sequence $HHTTHTHTT$, HT occurs 3 times.)

Solution

The expected number of Heads that will occur is $n\alpha$;
Let $F_1, F_2, F_3, \dots F_n$ denote the outcomes of the n flips and let's define the random variable X_i, where $i = 1, 2, 3, \dots n-1$ with $X_i = 1$ if $F_i = H$ and $F_{i+1} = T$. Then $X = X_1 + X_2 + \dots + X_{n-1}$ will be the number of times the sequence HT occurs, $E(X_i) = \alpha(1-\alpha)$ and $E(X) = E(X_1) + E(X_2) + \dots + E(X_{n-1}) = (n-1)\alpha(1-\alpha)$.