CSE21 WI13

Homework #3

[Each problem is worth 10 points.]

3.1 How many different 5-letter words (= strings) can be formed from the letters in CALCULUS if no letter can be used more times than it occurs in the word CALCULUS? For example, CCUUL and LSUCS would count but UUCSU would not count.

Answer: 2 C's, 2 U's, 2L's, 1 A, 1 S. 5 distinct letters: $\binom{0}{0}$ choices for letters with duplicates, $\binom{5}{5}$ choices for letters with no duplicates. 5! ways to arrange 5 unique letters for $\binom{0}{0}\binom{5}{5}5! = 5!$ ways.
4 distinct letters: $\binom{3}{1}$ choices for letters with duplicates, $\binom{4}{3}$ choices for letters with no duplicates. $\frac{5!}{2!}$ ways to arrange 5 letters with one pair of duplicates for $\binom{3}{1}\binom{4}{3}\frac{5!}{2!}$ ways.
3 distinct letters: $\binom{3}{2}$ choices for letters with duplicates, $\binom{3}{1}$ choices for letters with no duplicates. $\frac{5!}{2!2!}$ ways to arrange 5 unique letters with two duplicates for $\binom{3}{2}\binom{3}{1}\frac{5!}{2!2!}$ ways.
By the sum rule, this gives us $5! + \binom{3}{1}\binom{4}{3}\frac{5!}{2!} + \binom{3}{2}\binom{3}{1}\frac{5!}{2!2!}$ ways.

3.2 (a) For breakfast, you have a choice of 4 kinds of doughnuts: glazed, chocolate, sugar and plain. In how many different ways can you choose 5 of these doughnuts?

(b) What is the answer in the general case that there are k kinds of doughnuts and you want to select n doughnuts?

(Hint: think bars and stars.)

Answer: a) 3 bars, 5 stars for $\binom{8}{3}$ ways.
b) $k-1$ bars, n stars for $\binom{k+n-1}{n}$ ways.

3.3 A valid password is a 5 character string made up of letters {A,B, ..., Z} and numbers {0, 1, ... 9} such that **at least one** number and **at least**

two letters are used. How many valid passwords are there?

Answer: 36^5 possible passwords, 26^5 of which contain no number, 10^5 which of which contain no letter, and $\binom{5}{1}10^4 \cdot 26$ of which contain only one letter (5 possible positions for the number) for a total of $36^5 - 26^5 - 10^5 - \binom{5}{1}10^4 \cdot 26$ ways.

3.4 The generating function for the sequence $1, 1, 1, \ldots$ is $1 + x + x^2 + x^3 \ldots = \frac{1}{1-x}$. Write a formula for the generating function of each of the following sequences:

(i) $0, 0, 1, 1, 1, \ldots$

(ii) $1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots$

(iii) $1, 3, 5, 7, \ldots$

(iv) $1 \cdot 2,\ 2 \cdot 3,\ 3 \cdot 4,\ 4 \cdot 5, \ldots$.

Answer: (i) $\frac{x}{1-x} = \langle 1, 1, 1, ... \rangle$ so $\frac{x^2}{1-x} = \langle 0, 0, 1, 1, 1, ... \rangle$
(ii) $1 + x^3 + x^6 + ... = \frac{1}{1-x^3}$ (substitute x^3 for x)
(iii) $\frac{d}{dx}1 + x + x^2 + ... = 1 + 2x + 3x^2 + ...$ and $\frac{d}{dx}1 + x + x^2 + ... = \frac{1}{1-x}$ so $\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ...$ then $\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + ...$ and $\frac{1}{(1-x)^2} + \frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^2} = (1 + 0) + (2 + 1)x + (3 + 2)x^2 + (4 + 3)x^3 + ... = \langle 1, 3, 5, 7, ... \rangle$ Therefore our generating function is $\frac{1+x}{(1-x)^2}$
(iv) $\frac{d}{dx}1 + x + x^2 + ... = 1 + 2x + 3x^2 + ...$ and $\frac{d}{dx}1 + 2x + 3x^2 + ... = 1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + ...$ so the generating function we are looking for is $\frac{d^2}{dx^2}\frac{1}{1-x} = \frac{d}{dx}\frac{1}{(1-x)^2} = \frac{2}{(1-x)^3}$

3.5 We will use generating functions to determine how many ways there are to use pennies and nickels to give n cents change.

(i) Write the sequence P_n for the number of ways to use only pennies to change n cents. Write the generating function for that sequence.

(ii) Write the sequence N_n for the number of ways to use only nickels to change n cents. Write the generating function for that sequence.

(iii) Write the generating function for the number of ways to use pennies and nickels to change n cents.

(iv) Write the generating function for the number of ways to use pennies, nickels and dimes to change n cents.

(i) $\langle 1, 1, 1, 1, ... \rangle = \frac{1}{1-x}$
(ii) $\langle 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, ... \rangle = \frac{1}{1-x^5}$
(iii) $\frac{1}{1-x^5} \cdot \frac{1}{1-x}$ by the product rule
(iv) $\frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}}$ by the product rule

3.6 Suppose that the sequence $S = (s_0, s_1, s_2, \ldots)$ is defined by:
$s_0 = 0, s_1 = 1$ and $s_{n+2} = s_{n+1} + 2s_n$ for $n \geq 0$. Thus, $S = (0, 1, 1, 3, 5, 11, 21, \ldots)$.
Show that the generating function $S(x) = s_0 + s_1 x + s_2 x^2 + \ldots$ satisfies $S(x) = \frac{x}{1-x-2x^2}$.
(Hint: This is very similar to the Fibonacci number derivation we did in class (and also in the text)).

Answer:
$\langle 0, 1, 0, 0, 0, ... \rangle \leftrightarrow x$
$\langle 0, s_0, s_1 s_2, ... \rangle \leftrightarrow xS(x)$
$\langle 0, 0, 2s_0, 2s_1, 2s_2, ... \rangle \leftrightarrow 2x^2 S(x)$
so $\langle 0, 1 + 2s_0, s_1 + 2s_1, s_2 + 2s_1, ... \rangle \leftrightarrow x + xS(x) + 2x^2 S(x)$
then $S(x) \leftrightarrow \langle s_0, s_1, s_2, s_3, ... \rangle = \langle 0, 1 + 2s_0, s_1 + 2s_1, s_2 + 2s_1, ... \rangle \leftrightarrow x + xS(x) + 2x^2 S(x)$ so $S(x) = x + xS(x) + 2x^2 S(x)$
$S(x) - xS(x) - 2x^2 S(x) = x$
$S(x)(1 - x - 2x^2) = x$
$S(x) = \frac{x}{1-x-2x^2}$