CSE21 WI13

Homework #2

[Each problem is worth 10 points.]

2.1 Let S be a set of size $n > 0$. Show that the number of subsets of S with an *even* number of elements is equal to the number of subsets of S with an *odd* number of elements.
For example, for $S = \{a, b, c\}$, the even subsets are $\emptyset, \{a, b\}, \{a, c\}, \{b, c\}$ and the odd subsets are $\{a\}, \{b\}, \{c\}, \{a, b, c\}$.

Solution: By the binomial theorem, for $n > 0$,

$$0 = (1-1)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k} = \sum_{k\,\text{even}}\binom{n}{k} - \sum_{k\,\text{odd}}\binom{n}{k}.$$

In other words, $\sum_{k\,\text{even}}\binom{n}{k} = \sum_{k\,\text{odd}}\binom{n}{k}$, and since the right hand side corresponds to the number of subsets of S with an *odd* number of elements and the left hand side to the number of subsets of S with an *even* number of elements, we have completed our proof.

2.2 How many 5-card hands from an ordinary deck of 52 have all cards with the same color (i.e., all red = $\diamondsuit, \heartsuit$ or all black = $\clubsuit, \spadesuit$)?

Solution: There are 26 red cards and therefore $\binom{26}{5}$ all red 5-card hands, similarly there are $\binom{26}{5}$ all black, so by the sum rule a total of $2 \times \binom{26}{5}$ hands have all cards with the same color.

2.3 (a) How many paths are there from point (0,0) to (50,50) if every step increments one coordinate by one unit and leaves the other unchanged.

Solution: Each path from point (0,0) to (50,50) will contain 50 rightward steps (i.e. incremental along the x-axis) and 50 upward steps (incremental along y-axis), resulting in a total of 100 steps, we therefore have $\binom{100}{50}$ ways of creating such a path.

(b) How many are there when there are impassable boulders sitting at points (10,11) and (21,20)?

Since we can not go through the boulders, any path going through one of the points (10,11) and (21,20), or both, would have to be excluded from our count. Now, let's first count the paths through (10,11): Each consists of a partial path from (0,0) to (10,11) and then a path from (10,11) to (50,50), so by the product rule there are $\binom{21}{10}\binom{79}{40}$ paths of that kind. Similarly, there are $\binom{41}{21}\binom{59}{29}$ paths through (21,20). If we subtract these from the total number of paths, we will have double counted the paths going through both of (10,11) and (21,20), but the number of those is $\binom{21}{10}\binom{20}{11}\binom{59}{29}$ by the product rule. So, finally by the principle of inclusion/exclusion, the number of possible paths is $\binom{100}{50} - \binom{21}{10}\binom{79}{40} - \binom{41}{21}\binom{59}{29} + \binom{21}{10}\binom{20}{11}\binom{59}{29}$.

2.4 (a) How many ways are there of distributing 5 oranges and 7 bananas to 4 (distinct) students?

Applying the "bars and stars" method, we see that there are $\binom{8}{3}$ ways of distributing the oranges and $\binom{10}{3}$ ways of distributing the bananas among 4 students. By the product rule there are a total of $\binom{8}{3}\binom{10}{3}$ ways of distributing the fruit.

(b) What is the answer if each student must get at least one banana.?

We give each student one banana, then we have $\binom{6}{3}$ ways of distributing the remaining 3 bananas. The answer is therefore $\binom{8}{3}\binom{6}{3}$.

(c) (Extra credit) What is the answer if all you require is that each student gets at least one piece of fruit (orange or banana)?

There are four cases to consider, depending on how many of the students get a banana:

- All the students get at least one banana, giving us $\binom{8}{3}\binom{6}{3}$ ways to distribute the rest of the fruit (see 2.4 (b)).
- Three students get at least one banana, but one student none, and instead at least one orange. This amounts to $4\binom{7}{3}\binom{6}{2}$ more ways.
- The bananas are split between two students, but the other two get at least one orange. We count $\binom{4}{2}\binom{6}{3}\binom{6}{1}$ to do this.
- One students get all the bananas, but the remaining three each get

at least one orange. This results in $4\binom{5}{3}$ possibilities of assigning the bananas to one student, then distributing the remaining 2 oranges.

We finally have a total of $\binom{8}{3}\binom{6}{3} + 4\binom{7}{3}\binom{6}{2} + \binom{4}{2}\binom{6}{3}\binom{6}{1} + 4\binom{5}{3} = 3980$ ways.

Alternative solution: Let's count the complement. There are $\binom{7}{2}\binom{9}{2}$ ways to distribute the fruit, assuming we leave out a specific student. But if we multiply this with 4 (the number of students) we will have overcounted the cases where more than one students gets nothing. So, to compensate we use the principle of exclusion/inclusion to cover each case only once, and we get $\binom{8}{3}\binom{10}{3} - 4\binom{7}{2}\binom{9}{2} + 6\binom{6}{1}\binom{8}{1} - 4 = 3980$.

2.5 The working days in the next year can be numbered $1, 2, \ldots, 300$. I'd like to avoid as many as possible.

- On even-numbered days, I'll say I'm sick;
- On day that are a multiple of 3, I'll say I was stuck in traffic;
- On days that are a multiple of 5, I'll refuse to come out from under the blankets.

In total, how many work days will I *avoid* in the coming year?

Use the principle of inclusion/exclusion: Count the days that are multiples of 2, multiples of 3, multiples of 5, then subtract from this to account for the days that are multiples of 2 AND 3 (that is 6), 2 and 5, 3 and 5. By then we are still missing the days that are multiples of 2, 3 AND 5 (i.e multiples of 30), so we add these again. The answer then is $150+100+60-50-30-20+10 = 220$.

2.6 Show that in any set of 100 integers there is always some pair whose difference is divisible by 99.

Look at the remainders of the numbers modulo 99, i.e., remainders when you divide by 99. Two of the numbers must have the same remainder (pigeonhole principle). Their difference is divisible by 99.