CSE 21A

Practice Problem Solutions

March 12, 2013

The following problems are good practice for the Final Exam. It is very likely the almost all of the problems on the Final will be closely related to some of these problems. I have sketched solutions to them here.

1. If A and B are events in a probability space with $Pr(A) = \frac{1}{3}$, $Pr(B) = \frac{1}{4}$ and $Pr((A \cap B)^c) = \frac{11}{12}$, then what is $Pr((A \cup B)^c)$?

Solution. Since $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$ by inclusion/exclusion and we are given that $Pr((A \cap B)^c) = \frac{11}{12}$, $Pr(A) = \frac{1}{3}$, $Pr(B) = \frac{1}{4}$, then $Pr(A \cap B) = 1 - \frac{11}{12} = \frac{1}{12}$ so

$$Pr(A \cup B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{2}$$

and $Pr((A \cup b)^c) = 1 - Pr(A \cup B) = 1 - \frac{1}{2} = \frac{1}{2}$.

2. 8 Red and 9 Blue jellybeans are distributed randomly to 4 students. What is the probability that each student got *at least one* jellybean of each color?

Solution. By "Bars and Stars", the total number of ways that the Red jellybeans can be distributed is $\binom{8+3}{3}$ and the total number of ways that the Blue jellybeans can be distributed is $\binom{9+3}{3}$. Hence, the total number of ways that all the jellybeans can be distributed is (by the product rule) equal to $\binom{11}{3}\binom{12}{3}$. However, if we require that each student get at least one jellybean of each color, then we distribute these jellybeans (both Red and Blue) first, leaving just $8 - 4 = 4$ "free" Red jellybeans and $9 - 4 = 5$ "free" Blue jellybeans. Now, using Bars and Stars, we find that there are $\binom{4+3}{3}\binom{5+3}{3}$ ways to distribute these remaining jellybeans. Since we are assuming that all the distributions are equally likely, then the probability we want is

$$\frac{\binom{7}{3}\binom{8}{3}}{\binom{11}{3}\binom{12}{3}}.$$

3. How many 5-card hands can be formed from an ordinary deck of 52 cards if *exactly two* suits are present in the hand?

Solution. You have to be careful here not to overcount! There are $\binom{4}{2} = 6$ ways to choose the 2 suits. For example, suppose you choose $\spadesuit$'s and $\heartsuit$'s. Then possible hands can have

4♠'s and 1♡, or 3♠'s and 2♡'s, 2♠'s and 3♡'s, or 1♠ and 4♡'s, In the first case, there are $\binom{13}{4}\binom{13}{1}$ possible hands that have 4♠'s and 1♡. Computing the number of possible hands for the other combinations in the same way, we see that the total number of possible hands (by the sum rule) is $6(\binom{13}{4}\binom{13}{1} + \binom{13}{3}\binom{13}{2} + \binom{13}{2}\binom{13}{3} + \binom{13}{1}\binom{13}{4})$.

4. A shelf contains 24 books. How many ways can 6 books be selected from these 24 with the restriction that no two selected books can be adjacent?

Solution. This was the same kind of problem that we had on the first midterm. The idea is that you can think of each book *except* the right-hand most book as occupying **2** slots. Thus, there are really only $24 - 5 = 19$ places for these "fattened" books to occupy and so the number of choices is $\binom{19}{6}$. (If this isn't clear, try the same problem but with only 6 books on the shelf, and you are removing 3. In this case, the number of choices is $\binom{4}{3} = 4$.)

5. A hand H of 5 cards is chosen randomly from a standard deck of 52. Let E_1 be the event that H has *at least one* King and let E_2 be the event that H has *at least* 2 Kings. What is the conditional probability $Pr(E_2 \mid E_1)$?

Solution. The easy way to do this problem is to consider the following picture.

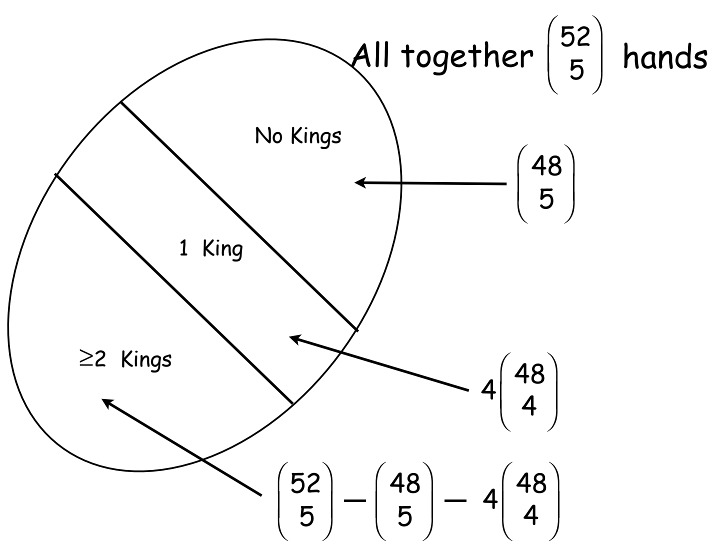


Figure 1: A partition of all $\binom{52}{5}$ hands.

Thus, the number of hands H with ≥ 1 King is $\binom{52}{5} - \binom{48}{5}$ and all these hands are equally likely. Among these, the number with *exactly one* King is $4\binom{48}{4}$. Therefore, $Pr(H \text{ has exactly one King} \mid H \text{ has} \geq 1 \text{ King}) = \frac{4\binom{48}{4}}{\binom{52}{5}-\binom{48}{5}}$.

What we want is the *complement*, namely
$Pr(H \text{ has} \geq 2 \text{ Kings} \mid H \text{ has} \geq 1 \text{ King}) = Pr(E_2 \,|\, E_1) = 1 - \frac{4\binom{48}{4}}{\binom{52}{5}-\binom{48}{5}}$.

A more brute force way of doing this problem is to count directly the number of various kinds of hands. In particular, the number of hands H which have exactly k Kings is just $\binom{4}{k}\binom{48}{5-k}$ for $k = 1, 2, 3$ and 4. Thus,

$$Pr(H \text{ has } \geq 1 \text{ King}) = \frac{1}{\binom{52}{5}} \sum_{k=1}^{4} \binom{4}{k}\binom{48}{5-k}.$$

Similarly,

$$Pr(H \text{ has } \geq 2 \text{ Kings}) = \frac{1}{\binom{52}{5}} \sum_{k=2}^{4} \binom{4}{k}\binom{48}{5-k}.$$

Thus,

$$Pr(E_2 \,|\, E_1) = \frac{Pr(H \text{ has } \geq 2 \text{ Kings and } H \text{ has } \geq 1 \text{ King})}{Pr(H \text{ has } \geq 1 \text{ King})} = \frac{Pr(H \text{ has } \geq 2 \text{ Kings})}{Pr(H \text{ has } \geq 1 \text{ King})} = \frac{\sum_{k=2}^{4}\binom{4}{k}\binom{48}{5-k}}{\sum_{k=1}^{4}\binom{4}{k}\binom{48}{5-k}}.$$

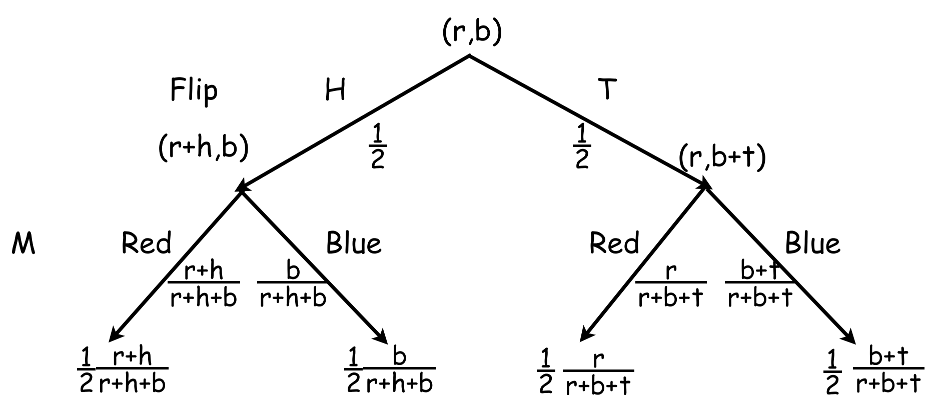
These two different looking expressions for $Pr(E_2 \,|\, E_1)$ have the same value (of course!).

6. An urn contains r Red and b Blue marbles. A fair coin is flipped. If the flip is Heads then h Red marbles are added to the urn. if the flip is Tails then t Blue marbles are add to the urn. Now a random marble M is drawn from the urn.

(a) What is the probability that M is Red?

(b) What is the probability that the flip was Heads given that M is Blue?

Solution. We construct the decision tree as shown below.



Thus, $Pr(M \text{ is Red}) = \frac{1}{2}\left(\frac{r+h}{r+h+b} + \frac{r}{r+b+t}\right)$.

Also, we have $Pr(\text{flip is Heads} \mid M \text{ is Blue}) = \frac{\frac{b}{r+h+b}}{\frac{b}{r+h+b}+\frac{b+t}{r+b+t}}$.

(Question: Is it easier using *letters* rather than *numbers* for the values r, b, h, t?)

7. A fair coin is flipped 3 times resulting in the flip sequence $F_1F_2F_3$.. Consider the three events:
(i) E_1 – $\{F_1 \text{ is Heads}\}$;
(ii) E_2 – $\{F_2 \text{ and } F_3 \text{ agree}\}$;
(iii) E_3 – $\{F_1 \text{ and } F_3 \text{ disagree.}\}$

Which of the 3 pairs of events are *independent?*

Solution. *All three* pairs are independent. To see this, it is enough to tabulate the E_i:
$E_1 = \{HHH, HHT, HTH, HTT\}, E_2 = \{HHH, THH, HTT, TTT\}, E_3 = \{HHT, HTT, THH, TTH\}$.
Thus, $E_1 \cap E_2 = \{HHH, HTT\}$, $E_1 \cap E_3 = \{HHT, HTT\}$, $E_2 \cap E_3 = \{HTT, THH\}$.
Consequently, all the E_i have probability $\frac{1}{2}$ and all the intersections have probability $\frac{1}{4}$. So, since $Pr(E_i \cap E_j) = \frac{1}{4} = Pr(E_i)Pr(E_j)$ for $i \neq j$, then all the pairs are independent.

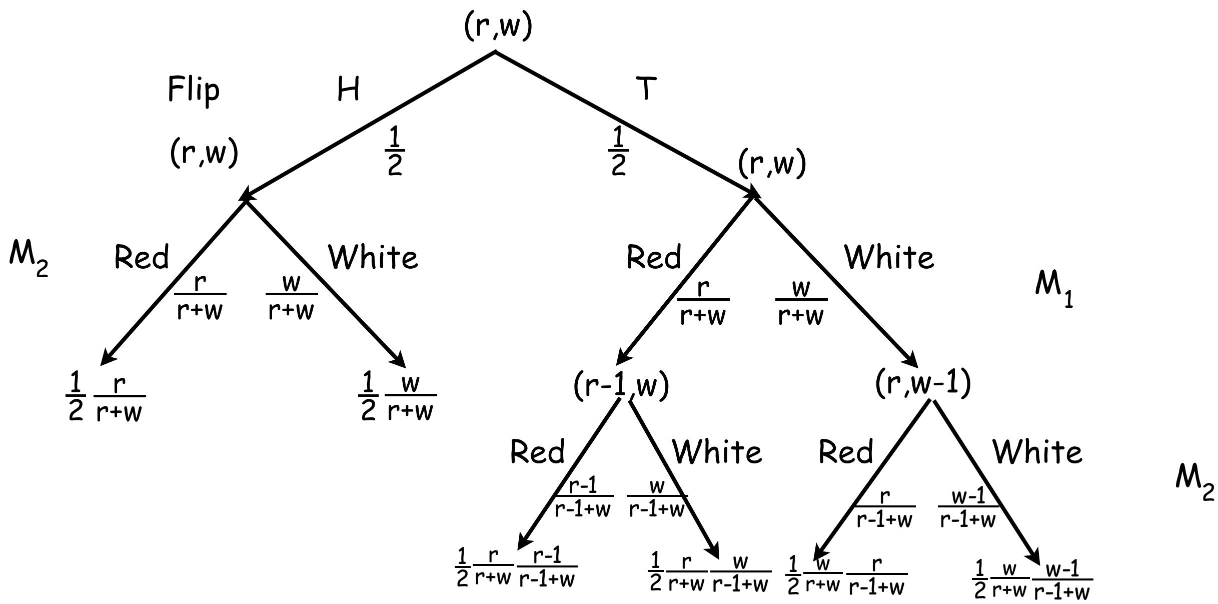
8. A biased coin C has $Pr(H) = \alpha$ and $Pr(T) = 1 - \alpha$ (where $0 \leq \alpha \leq 1$). The coin C is flipped n times. What is the expected number of times that the consecutive sequence HXH occurs where X can be either H or T? (For example, if the flip sequence were $HHTHHHTHTHHT$ then this number would be 4.)

Solution. Let X denote the random variable which counts the number of occurrences of HXH in our flip sequence $F_1F_2\ldots F_n$. We use the standard trick of decomposing $X = X_1 + X_2 + \ldots + X_{n-2}$ where $X_i = 1$ if $F_iF_{i+1}F_{i+2} = HXT$ (where X can be either H or T), and $X_i = 0$, otherwise. Now by *linearity of expectation*, $\mathbf{E}(X) = \mathbf{E}(X_1) + \mathbf{E}(X_2) + \ldots \mathbf{E}(X_{n-2})$. It is easy to calculate that $\mathbf{E}(X_i) = \alpha^2$ (which is just the probability that $X_i = HXH$). Therefore, $\mathbf{E}(X) = (n-2)\alpha^2$.

9. An urn contains r Red and w White marbles. A random marble M_1 is drawn and a fair coin is flipped. If the flip is Heads then M_1 is put back into the urn. On the other hand, if the flip is Tails, the marble M_1 is *not* put back into the urn. Now another random marble M_2 is drawn from the urn.

(i) What is $Pr(M_2 = \text{Red})$?
(ii) What is $Pr(M_1 = \text{Red} \mid M_2 = \text{Red})$?
(iii) What is $Pr(\text{Flip is Heads} \mid M_2 = \text{White})$?

There are two ways (at least!) to tackle this problem. The first way is a brute force approach which we will now demonstrate. Let's first construct the decision tree for this process (which we show below).



Now it is just a matter of reading off the appropriate probabilities from the decision tree. So, $Pr(M_2 = \text{Red}) = \frac{1}{2}\frac{r}{r+w} + \frac{1}{2}\frac{r}{r+w}\frac{r-1}{r-1+w} + \frac{1}{2}\frac{w}{r+w}\frac{r}{r-1+w}$. However, this expression simplifies to give $Pr(M_2 = \text{Red}) = \frac{r}{r+w}$! (We'll see a reason for this when we use a more clever way to solve the problem). Similarly, we get $Pr(M_1 = \text{Red} \,|\, M_2 = \text{Red}) = \frac{Pr(M_1 = \text{Red and } M_2 = \text{Red})}{Pr(M_2 = \text{Red})} = \frac{\frac{1}{2}\frac{r}{r+w}\frac{r-1}{r-1+w}}{\frac{r}{r+w}} = \frac{1}{2}\frac{r-1}{r-1+w}$. Finally, since $Pr(M_2 = \text{ White}) = 1 - \frac{r}{r+w} = \frac{w}{r+w}$ then $Pr(\text{Flip is Heads} \,|\, M_2 = \text{White}) = \frac{1}{2}$.

However, there is a better way to look at this problem (we also did this in class). This uses the fact that when the flip was Tails and we drew out the random marble M_1 from the urn, we didn't do anything in the process that depended on its color. So we could have just as well imagined that in this case we drew out *two* random marbles after the flip and then chose one of them at random to be M_2. So, in this case (when the flip was Tails), the probability that M_2 was Red was just $\frac{r}{r+w}$, just as in the case then the flip was Heads. Thus, $Pr(M_2 = \text{Red}) = \frac{1}{2}\frac{r}{r+w} + \frac{1}{2}\frac{r}{r+w} = \frac{r}{r+w}$. More generally, the probabilities of the other events in the problem are just the same as if we had never drawn out M_1 (since it was drawn out randomly and we never use its color). This explains how we get $Pr(M_1 = \text{Red} \,|\, M_2 = \text{Red}) = \frac{1}{2}\frac{r-1}{r-1+w}$ and $Pr(\text{Flip is Heads} \,|\, M_2 = \text{White}) = \frac{1}{2}$.

10. How many different way are there of arranging **all** the letters of the string **CALCULUSBOOK**?

Solution. We use the "BOOKEEPER" rule from the syllabus. The 12 letters from our word are $ABC^2KL^2O^2SU^2$. Thus, the number of rearrangements is $\frac{12!}{2!2!2!2!}$.

11. What is the coefficient of x^5 in the expansion of $(3x-1)^{11}$?

Solution. By the binomial theorem, the coefficient of x^k in the expansion of $(ax+b)^n$ is $\binom{n}{k}a^kb^{n-k}$. Thus, for our problem, the desired coefficient is $\binom{11}{5}3^5(-1)^6 = \binom{11}{5}3^5$.

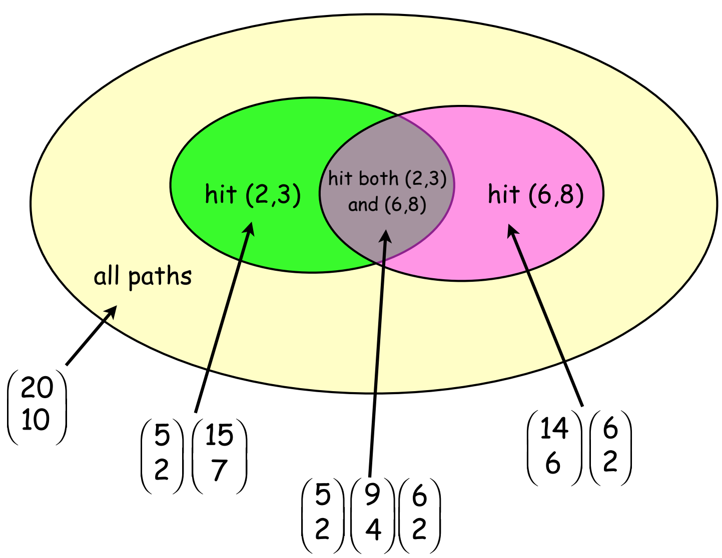
12. We want to count step-by-step paths between points with integer coordinates. Only two kinds of steps are allowed: a right-step which increments the x coordinate by 1 and an up-step which increments the y coordinate by 1.

(i) How many paths are there from point $(0,0)$ to point $(10,10)$?

(ii) How many paths are there if there is an impassable boulder sitting at point $(5,6)$?

(iii) How many paths are there if there are impassable boulders sitting at points $(2,3)$ and $(6,8)$?

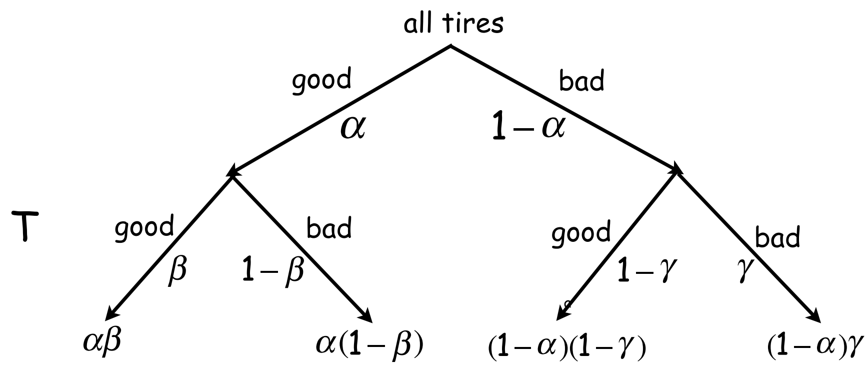
Solution. For part (i), our path must take 10 up-steps and 10 right-steps, and any order is possible. Thus, the number of choices is $\binom{20}{10}$ (which is a pretty big number!).
For part (ii), we first want to compute the *complement*, namely how many paths go through the point $(5,6)$? Since there are $\binom{11}{5}$ paths from $(0,0)$ to $(5,6)$ and $\binom{9}{4}$ paths from $(5,6)$ to $(10,10)$ then by the product rule, there are $\binom{11}{5}\binom{9}{4}$ paths from $(0,0)$ to $(10,10)$ which go through $(5,6)$. Hence, the number which *don't* go through $(5,6)$ is $\binom{20}{10} - \binom{11}{5}\binom{9}{4}$.
Part(iii) is a little trickier, and uses inclusion/exclusion. The figure below is a Venn diagram showing the various sets of paths. The number we want is the number of that don't hit any of



the points $(2,3)$ or $(6,8)$, which by inclusion/exclusion is $\binom{20}{10} - \binom{5}{2}\binom{15}{7} - \binom{14}{6}\binom{6}{2} + \binom{5}{2}\binom{9}{4}\binom{6}{2}$.

13. The Acme Tire company manufactures tires. It is known that with probability α a randomly selected tire is good, and with probability $1-\alpha$, a randomly selected tire is bad. There is a test T which behaves as follows. If T is applied to a good tire then with probability β, it says that the tire is good (so with probability $1-\beta$, T says that the tire is bad). On the other hand, if T is applied to a bad tire then with probability γ, it says that the tire is bad (and with probability $1-\gamma$, it says that the tire is good).
What is the probability that a randomly selected tire is good given that the test T says that it is bad?

Solution. As usual, we first construct the decision tree, as shown below.



Hence, the probability we want is

$$Pr(\text{tire is good} \mid \text{test says bad}) = \frac{Pr(\text{tire is good } \textit{and} \text{ test says bad})}{Pr(\text{test says bad})} = \frac{\alpha(1-\beta)}{\alpha(1-\beta)+(1-\alpha)\gamma}.$$

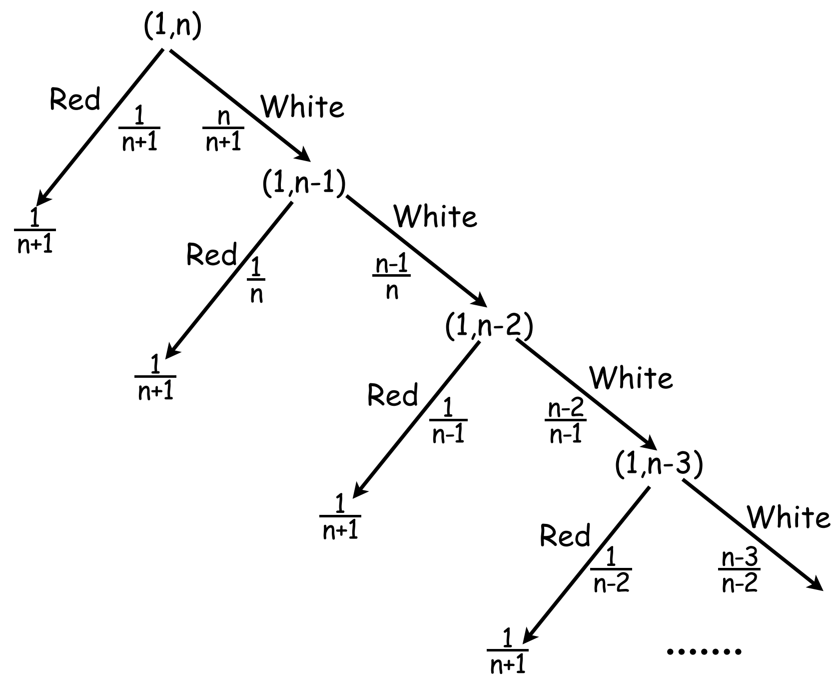
14. The generating function for the sequence $\langle 1,1,1,1,\ldots\rangle$ is $1+x+x^2+x^3\ldots = \frac{1}{1-x}$.
What is the generating function for the sequence $\langle 1,1,3,3,5,5,7,7,9,9,\ldots\rangle$?

Solution. The generating function for $\langle 0,0,1,1,1,1,\ldots\rangle$ (which is $\langle 1,1,1,1,\ldots\rangle$ shifted to the right by two places) is just $\frac{x^2}{1-x}$. More generally, the generating function for shifting $\langle 1,1,1,1,\ldots\rangle$ to the right by $2k$ places (to get $\langle 0,0,\ldots 0,1,1,1,1,\ldots\rangle$ with $2k$ 0's) is $\frac{x^{2k}}{1-x}$. Multiplying by 2, we see that the generating function for $\langle 0,0,\ldots 0,2,2,2,2,\ldots\rangle$ with $2k$ 0's) is just $\frac{2x^{2k}}{1-x}$. Adding all these up, we find that the generating function for $\langle 1,1,3,3,5,5,7,7,9,9,\ldots\rangle$ is $\frac{1}{1-x}(1+2\sum_{k\geq 1} x^{2k}) = \frac{1}{1-x}(1+\frac{2x^2}{1-x^2}) = \frac{1+x^2}{(1-x)(1-x^2)}$.

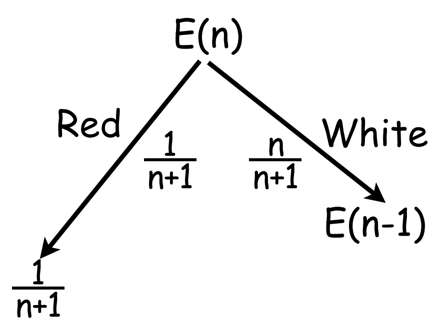
15. An urn contains 1 Red ball and n White balls. You repeatedly draw balls *without replacement* until you get the Red ball. What is the expected number of draws until this happens?

What is the answer if instead you draw *with replacement*?

Solution. Let us draw the decision tree, as shown below.



Notice the interesting fact that the probabilities at each of the terminal Red branches on the left are all equal to $\frac{1}{n+1}$ (as we take the products of the probabilities going down to the end of the branch). Since the first branch takes just 1 step, and the second branch takes 2 steps, and in general, the k^{th} branch takes k steps, and there are altogether $n+1$ branches, then the expected value for the number of steps taken is just $\frac{1}{n+1}\sum_{k=1}^{n+1} k = \frac{1}{n+1}\binom{n+2}{2} = \frac{n+2}{2}$.
A more compact of seeing this is using the following "recursive" decision tree shown below. Here, we let $E(m)$ denote the expected number of steps required if we have m White balls and 1 Red ball. Thus, $E(0) = 1$.



Thus, we have the recurrence $E(n) = 1 \cdot \frac{1}{n+1} + (1 + E(n-1)) \cdot \frac{n}{n+1}$ (where the +1 in the

factor $(1+E(n-1)$ comes from the fact that we took 1 additional step to get to that point). Multiplying both sides by $n+1$ and substituting $F(n) = (n+1)E(n)$ for all n, we have the simple (!) recurrence for F, namely $F(n) = n + 1 + F(n-1)$, $F(0) = 1 \cdot E(0) = 1$. The solution to this recurrence is $F(n) = \binom{n+2}{2}$ so we find that $E(n) = \frac{1}{n+1}\binom{n+2}{2} = \frac{n+2}{2}$.
For the case when we draw with replacement, the decision tree is actually *infinite*. However, with $E^*(n)$ denoting the expected number of steps taken in this case, then the same recursive analysis as above gives the recurrence $E^*(n) = 1 \cdot \frac{1}{n+1} + (1 + E^*(n)) \cdot \frac{n}{n+1}$. However, this directly implies that $E^*(n) = n + 1$. Thus, we can expect to wait about twice as long when drawing with replacement compared to drawing without replacement. An interesting problem is to see what happens in these two cases when we start with 2 (or k) Red balls and n White balls.

16. A sequence is defined by: $a(1) = 1$ and $a(n+1) = 3a(n) - 1$ for $n \geq 1$. What is $a(100)$?

Solution. The first few terms of $a(n)$ are $1, 2, 5, 14, \ldots$. The *general* solution to the homogeneous form of the recurrence is $a(n) = c \cdot 3^n$. A specific (constant) solution α must satisfy $\alpha = 3\alpha - 1$, so we find $\alpha = \frac{1}{2}$. Now, assuming the general solution to the complete recurrence has the form $a(n) = c \cdot 3^n + \frac{1}{2}$, and substituting $n = 1$, we see that $a(1) = 1 = 3c + \frac{1}{2}$ so that $c = \frac{1}{6}$. Thus, the solution we want is $a(n) = \frac{1}{6} \cdot 3^n + \frac{1}{2} = \frac{1}{2}(3^{n-1} + 1)$. (Check that this does generate the first few terms of $a(n)$). In particular, $a(100) = \frac{1}{2}(3^{99} + 1)$.

17. How many sequences of length n made up of 1, 2 and 3 do not have *two consecutive repeated* symbols?
(For example, 012120210212 would be allowed but 012112021021 would not.)

Solution. Observe that at any point in generating such a sequence, say we have so far $s_1 s_2 \ldots s_t$, then there are exactly *two* choices for the next symbol s_{t+1}. Since any of $1, 2$ or 3 can be the starting symbol, then the total number of sequences of length n is $3 \cdot 2^{n-1}$.

18. What is the general solution to the recurrence: $t(n+2) = 2t(n+1) + 2t(n)$, $n \geq 0$, with $t(0) = 0, t(1) = 1$?

Solution. The first few values of $t(n)$ are $0, 1, 4, 10, 28, \ldots$. The auxiliary equation for the recurrence is $r^2 = 2r + 2$, which has the two roots $r_1 = 1 + \sqrt{3}$ and $r_2 = 1 - \sqrt{3}$. Thus, the general solution to the recurrence has the form $t(n) = c_1(1+\sqrt{3})^n + c_2(1-\sqrt{3})^n$. Plugging in the values $t(0) = 0 = c_1 + c_2$ and $t(1) = 1 = c_1(1+\sqrt{3}) + c_2(1-\sqrt{3})$ and solving for the c_i gives the final solution $t(n) = \frac{1}{2\sqrt{3}}((1+\sqrt{3})^n - (1-\sqrt{3})^n)$.

19. What is the general solution to the recurrence: $x(n+2) = 3x(n+1) - 2x(n) + n$, $n \geq 0$, with $x(0) = 0, x(1) = 1$? (Hint: Try a *quadratic* polynomial for the specific solution to the recurrence.)

Solution. The first few values of $x(n)$ are $0, 1, 3, 8, 20, \ldots$. The auxiliary equation for the recurrence is $r^2 - 3r + 2$ which conveniently factors as $(r-1)(r-2)$. Hence, the roots are 1 and 2 and the general solution to the homogenous form of the recurrence is $x(n) = c_1 + c_2 2^n$ (since $1^n = 1$). The next step is to find some solution to the full inhomogeneous recurrence (i.e., with the additional term $+n$ in it). Suppose we try a solution of the form $\alpha n + \beta$. Then we must have $\alpha(n+2) + \beta = 3\alpha(n+1) + 3\beta - 2\alpha n - 2\beta + n$ which should hold for all values of n. But this implies that the coefficient of n which is $\alpha - 3\alpha - 2\alpha - 1 = -1$ must be *zero*, which it certainly isn't! So let us try a quadratic solution of the form $\alpha n^2 + \beta n + \gamma$. Plugging this into the recurrence gives $\alpha(n+2)^2 + \beta(n+2) + \gamma = 3\alpha(n+1)^2 + 3\beta(n+1) + 3\gamma - 2\alpha n^2 - 2\beta n - 2\gamma + n$. Collecting the coefficients of powers of n, we get $0 \cdot n^2 + (4\alpha + \beta - 6\alpha - 3\beta + 2\beta - 1)n + (4\alpha + 2\beta + \gamma - 3\alpha - 3\beta - 3\gamma + 2\gamma) = 0$. Since this must hold for all values of n, then the coefficients of n^2, n and the constant term must all be zero. Solving for α and β gives $\alpha = \beta = -\frac{1}{2}$ (there is no restriction on γ). So the general solution to the complete recurrence has the form $x(n) = c_1 + c_2 2^n - \frac{1}{2}(n^2 + n)$. Now plugging the boundary values $x(0) = 0 = c_1 + c_2$ and $x(1) = 1 = c_1 + 2c_2 - 1$, and solving for the c_i, we find that $c_1 = -2, c_2 = 2$ and consequently $x(n) = 2^{n+1} - \frac{1}{2}(n^2 + n) - 2$. (Definitely check a few initial values to reassure yourself, or prove by induction that this answer is correct).

20. What is the general solution to the recurrence: $x(n+2) = 6x(n+1) - 9x(n)$, $n \geq 0$, with $x(0) = 0, x(1) = 1$?

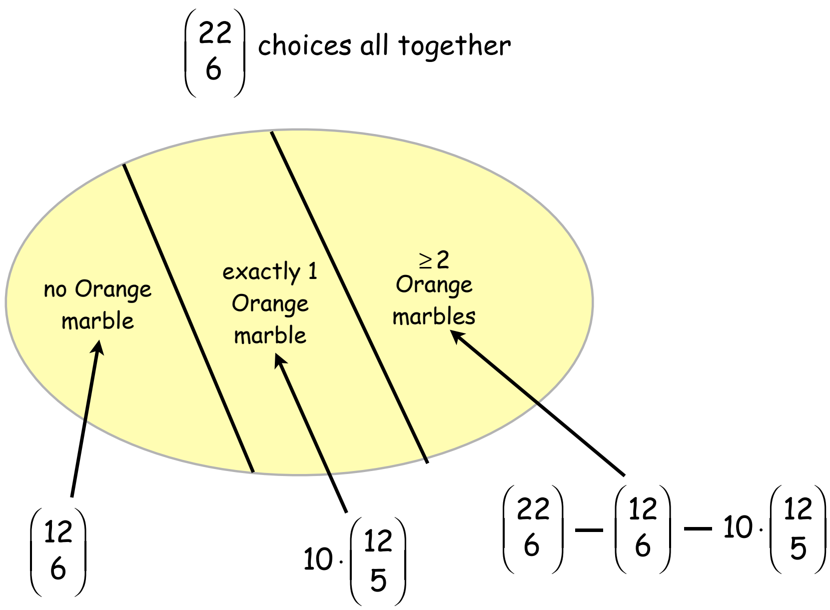
Solution. The first few values of $x(n)$ are $0, 1, 6, 27, \ldots$. The auxiliary equation for the recurrence is $r^2 - 6r + -9$ which factors as $(r-3)(r-3)$. Thus, we have *repeated roots* so that the general solution to the recurrence will have the form $x(n) = c_1 3^n + c_2 n 3^n$. Substituting the values $n = 0$ and $n = 1$ gives us $x(0) = 0 = c_1$ and $x(1) = 1 = 0 + c_2 \cdot 3$. Thus, $c_2 = \frac{1}{3}$ and $x(n) = n3^{n-1}$.

21. An urn contains 6 Red balls and 1 Blue ball. A fair die having faces $\{1, 2, 3, 4, 5, 6\}$ is rolled. If the top face on the die shows m, then m random balls are removed from the urn. What is the expected number of Red balls removed by this process?

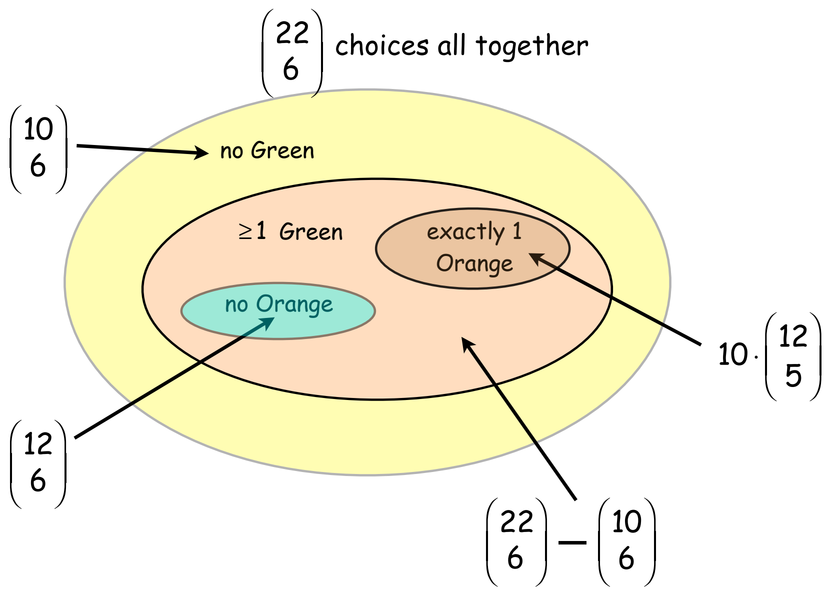
Solution. Since the die is fair, each one of the faces has probability $\frac{1}{6}$ of ending up on top. Suppose the value k shows. What is the expected number of Red balls obtained when we draw out k at random without replacement from our urn? Let us assume that we are drawing balls out one at a time so that our draw sequence is $(B_1, B_2, \ldots, B_k)$. Define a random variable on this sample space by defining $X(B_1, B_2, \ldots, B_k) = \#$ of Red balls drawn. As usual, decompose $X = X_1 + X_2, \ldots + X_k$ where $X_i(B_1, B_2, \ldots, B_k) = 1)$ if $B_i =$ Red, and 0 if B_i is Blue. Clearly, $(E)(X_i) = \frac{6}{7}$ so by linearity of expectation, $\mathbf{E}(X) = \frac{6}{7}k$. Since each one of the values of k occurs with probability $\frac{1}{6}$, then the expected number of Red balls drawn by this process is $\frac{1}{6}\sum_{k=1}^{6}\frac{6}{7}k = \frac{1}{6}\frac{6}{7}\binom{7}{2} = 3$.

22. A bin contains contains 10 Orange and 12 Green marbles. A random set S of 6 marbles are removed from the bin *without replacement*. What is the probability that S contains *at least* 2 Orange marbles, given that S contains an Orange marble? What is the probability that that S contains *at least* 2 Orange marbles, given that S contains a Green marble?

Solution. This problem is very similar to Problem 5. Let's do it the easy way. Consider the various sets shown in the figure below. Thus, since there are just $\binom{22}{6} - \binom{10}{6}$ ways of



choosing 6 marbles with at least one Orange marble, then the probability of choosing 6 marbles with exactly one Orange given that the selection is known to have at least one Orange is $\frac{10\binom{12}{5}}{\binom{22}{6}-\binom{12}{6}}$. Therefore, the complementary event, namely that the selection has at least two Orange marbles, given that it is known that it has at least one Orange is $1 - \frac{10\binom{12}{5}}{\binom{22}{6}-\binom{12}{6}}$.
For the second part we want $Pr(S \text{ has } \geq 2 \text{ Orange} \mid S \text{ has} \geq 1 \text{ Green})$. Consider the Venn diagram below.



The desired probability is seen to be $\frac{\binom{22}{6}-\binom{10}{6}-\binom{12}{6}-10\binom{12}{5}}{\binom{22}{6}-\binom{10}{6}}$.

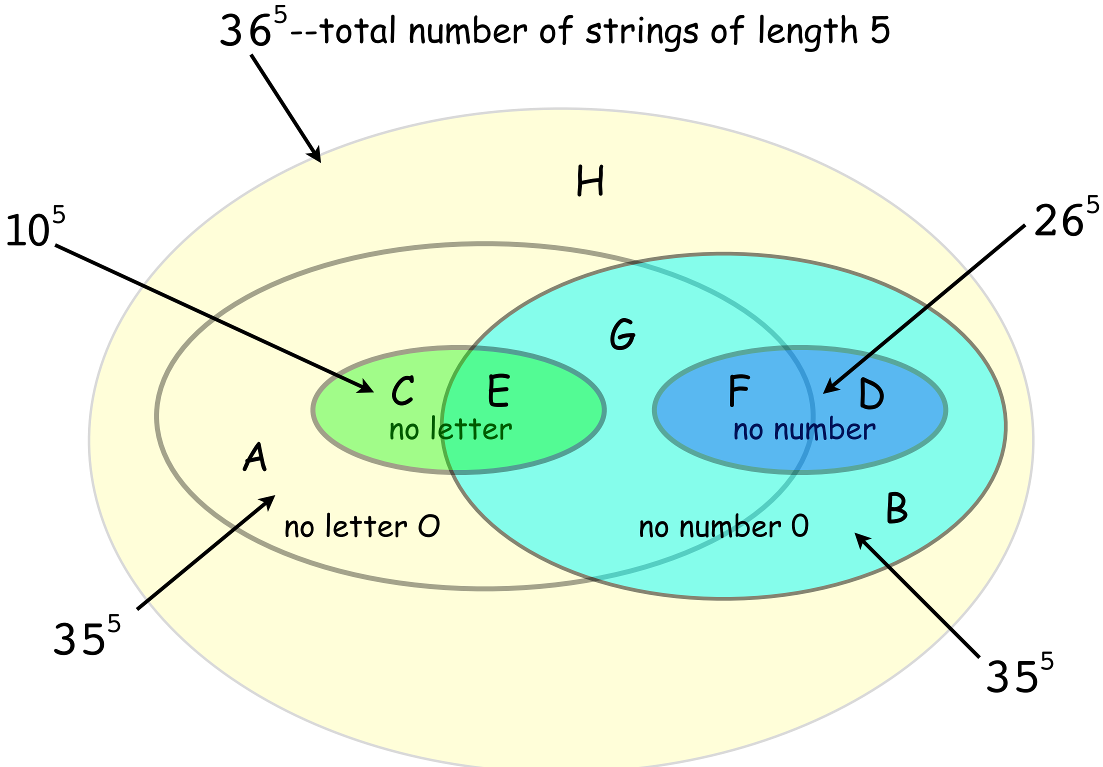
23. In how many ways can 10 identical cookies be distributed to 2 boys and 3 girls if no boy gets more than 1 cookie and every girl gets at least 1 cookie?

Solution. This is Bars and Stars with a twist. First, let's distribute the 3 cookies to the girls at the beginning, leaving 7 "free" cookies to distribute. Now, we consider 4 cases which account for which boys get cookies or not. Namely, if neither boy gets a cookie, then there are $\binom{7+2}{2}$ ways to distribute the 7 free cookies (3 girls implies 2 Bars). On the other hand, if just one of the boys gets a cookie, then there are 6 cookies left to distribute to the girls and this can be done in $\binom{6+2}{2}$ ways (but don't forget that there two ways for this to happen, namely Boy_1 gets a cookie and Boy_2 doesn't, or the other way around). Finally, if both boys get cookies, then there are only 5 cookies left to distribute to the girls and this can be done in $\binom{5+2}{2}$ ways. Thus, the total number of ways is $\binom{9}{2} + 2\binom{8}{2} + \binom{7}{2}$.

24. A valid password P consists of 5 characters taken from the sets of 26 letters $\{A, B, C, \ldots, Z\}$ and 10 numbers $\{0, 1, 2, \ldots, 9\}$. However, P must have at least one number and at least one number, and furthermore, P cannot have both of the symbols O and 0 in it. How many valid passwords are there?

Solution. This problem was slightly hairier than I planned! Consider the following Venn diagram of the various sets. It is divided up into 8 *disjoint* regions formed by 4 ovals.

For example, the union of regions A, C, E, G and F, which I'll denote by $ACEGF$, consists



of all the passwords which are missing the letter O. Thus, the cardinality of $ACEGF$ is 35^5. Similarly, $BDEGF$ consists of all the passwords which are missing the number 0, and so also has cardinality $|BDEGF| = 35^5$. Furthermore, the passwords in EGF are missing both the letter O and the number 0, and so $|EGF| = 34^5$. Thus, there are $2 \cdot 35^5 - 34^5$ passwords in $ABCDEFG$ by inclusion/exclusion. Also, region CE has all the passwords which have no letter so that $|CE| = 10^5$. Similarly, DF has all the passwords which have no number so that $|DF| = 26^5$. Hence the number of passwords satisfying our conditions is just $2 \cdot 35^5 - 34^5 - 10^5 - 26^5$. (It is a good thing we didn't have requirements on the number of upper case and lower case letters as well!).

25. What is the length of the Minimum Spanning Tree for the following weighted graph?

Solution. The length of any minimum spanning tree for this graph (and there is more than one) is 60.

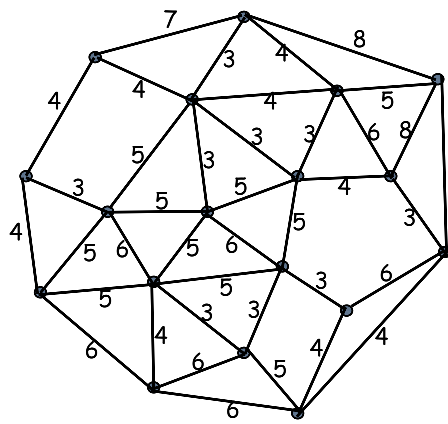


Figure 2: A weighted graph G.

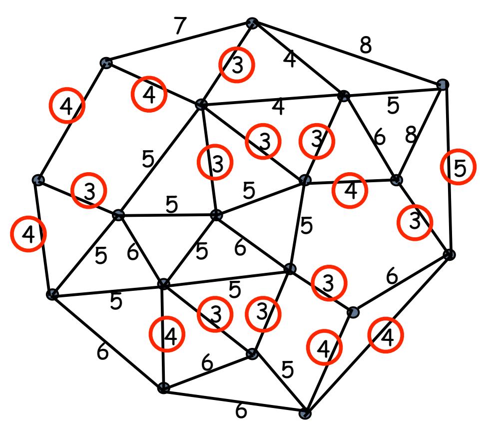


Figure 3: A minimum spanning tree for G