Random Variables
and
Random Vectors

Good Review Materials

http://www.imageprocessingbook.com/DIP2E/dip2e_downloads/review_material_downloads.htm

- (Gonzales & Woods review materials)

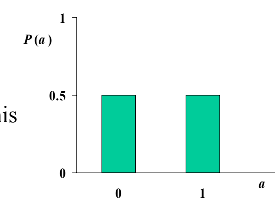
- Chapt. 1: Linear Algebra Review
- Chapt. 2: Probability, Random Variables, Random Vectors

Random variables

- Samples from a random variable are real numbers
  - A random variable is associated with a probability distribution over these real values
  - Two types of random variables
    - Discrete
      - Only finitely many possible values for the random variable: $X \in \{a_1, a_2, \ldots, a_n\}$
      - (Could also have a countable infinity of possible values)
        - » e.g., the random variable could take any positive integer value
      - Each possible value has a finite probability of occurring.
    - Continuous
      - Infinitely many possible values for the random variable
      - E.g., $X \in$ {Real numbers}

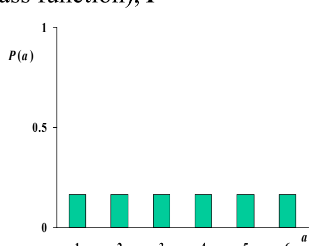
Discrete random variables

- Discrete random variables have a pmf (probability mass function), $\boldsymbol{P}$
  $P(X = a) = \boldsymbol{P}(a)$
- Example: Coin flip
  X = 0 if heads
  X = 1 if tails
  - What is the pmf of this random variable?



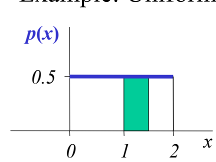
Discrete random variables

- Discrete random variables have a pmf (probability mass function), $\boldsymbol{P}$
  $P(X = a) = \boldsymbol{P}(a)$
- Example: Die roll
  $X \in \{1, 2, 3, 4, 5, 6\}$
  - What is the pmf of this random variable?



Continuous random variables

- Continuous random variables have a pdf (probability density function), $\boldsymbol{p}$
- Example: Uniform distribution



$p(1.3) = ?$ $p(2.4) = ?$

What is the probability that $X = 1.3$ exactly:
$P(X = 1.3) = ?$

Probability corresponds to area under the pdf.

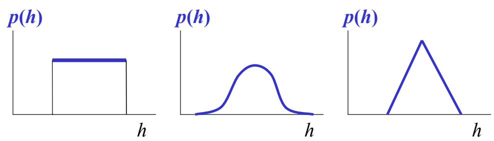
$$P(1 < X < 1.5) = \int_1^{1.5} p(x)\,dx = 0.25$$

Continuous random variables

- What is the total area under any pdf ?

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

- Example continuous random variable: Human heights



Random variables

- How much change do you have on you?
  – Asking a person (chosen at random) that question can be thought of as sampling from a random variable.

- Is the random variable "Amount of change people carry" discrete or continuous?

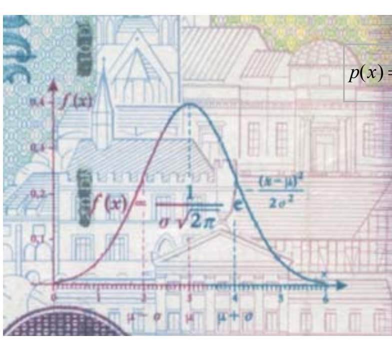
Random variables: Mean & Variance

- These formulas can be used to find the mean and variance of a random variable when its true probability distribution is known.

	Definition	Discrete r.v.	Continuous r.v.
Mean μ	$\mu = \mathrm{E}(X)$	$\mu = \sum_i a_i\, P(a_i)$	$\mu = \int_{-\infty}^{\infty} x\, p(x)\, dx$
Variance Var(X)	$\mathrm{E}\left((X-\mu)^2\right)$	$\sum_i (a_i-\mu)^2\, P(a_i)$	$\int_{-\infty}^{\infty} (x-\mu)^2\, p(x)\, dx$

An important type of random variable

The Gaussian distribution



$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$X \sim \mathrm{N}(\mu, \sigma^2)$$

Estimating the Mean & Variance

– After sampling from a random variable n times, these formulas can be used to estimate the mean and variance of the random variable.

- Samples $x_1, x_2, x_3, \ldots, x_n$

Estimated mean: $m = \frac{1}{n}\sum_{i=1}^{n} x_i$

Estimated variance: $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - m)^2$ ← maximum likelihood estimate

$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - m)^2$ ← unbiased estimate

Finding mean, variance in Matlab

– Samples $x = [x_1 \;\; x_2 \;\; x_3 \;\; \cdots \;\; x_n]$

– Mean

```
>> m = (1/n)*sum(x)
```

– Variance

$$\sigma^2 = \frac{1}{n}[x_1 - m \quad x_2 - m \quad \cdots \quad x_n - m]\begin{bmatrix} x_1 - m \\ x_2 - m \\ \vdots \\ x_n - m \end{bmatrix}$$

Method 1:
```
>> v = (1/n)*(x-m)*(x-m)'
```
Method 2:
```
>> z = x-m
>> v = (1/n)*z*z'
```

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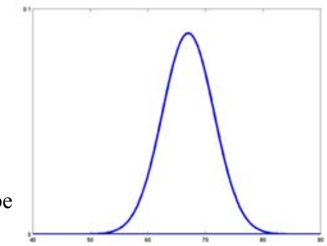
Example continuous random variable

- People's heights (made up)
  - Gaussian

    $\mu = 67$, $\sigma^2 = 20$



- What if you went to a planet where heights Gaussian

    $\mu = 75$, $\sigma^2 = 5$

  - How would they be different from us?

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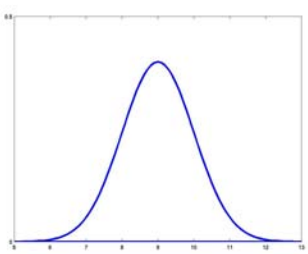
Example continuous random variable

- Time people woke up this morning
  - Gaussian

    $\mu = 9$, $\sigma^2 = 1$



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Random vectors

– An n-dimensional **random vector** consists of n random variables that are all associated with the same events.

– Example 2-D random vector:

$$\mathbf{V} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

where X is random variable of human **heights**

Y is random variable of **wake-up times**

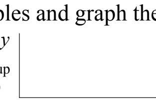
– Sample n times from V.

$$\begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{array}$$
$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

Let's collect some samples and graph them:

y

(wake-up times)



x (heights)

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Random Vectors

- What will the graph of V look like?

- What is mean of V ?
  - Mean of X is 67
  - Mean of Y is 10

$$m = \begin{bmatrix} 67 \\ 10 \end{bmatrix}$$

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Mean of a random vector

- Estimating the mean of a random vector
  - n samples from V

$$\begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{array}$$
$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

Mean $$\mathbf{m} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{v}_i = \frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} m_x \\ m_y \end{bmatrix}$$

– To estimate mean of V in Matlab

```
>> (1/n)*sum(v,2)
```

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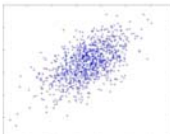
Random vector

– Example 2-D random vector:

$V = \begin{bmatrix} X \\ Y \end{bmatrix}$ where X is random variable of human **heights**
Y is random variable of human **weights**

- Sample n times from V $\quad \begin{matrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{matrix}$

– What will graph look like? $\quad \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$



Covariance of two random variables

- Height and wake-up time are uncorrelated, but height and weight are correlated.
- Covariance

$\mathrm{Cov}(X, Y) = 0$ for X = height, Y = wake-up times

$\mathrm{Cov}(X, Y) > 0$ for X = height, Y = weight

– Definition:

$$\mathrm{Cov}(X,Y) = \mathrm{E}\big((X-\mu_x)(Y-\mu_y)\big)$$

If $\mathrm{Cov}(X, Y) < 0$ for two random variables X, Y, what would a scatterplot of samples from X, Y look like?

Estimating covariance from samples

- Sample n times: $\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$

$$Cov(X,Y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - m_x)(y_i - m_y) \quad \leftarrow \text{maximum likelihood estimate}$$

$$Cov(X,Y) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - m_x)(y_i - m_y) \quad \leftarrow \text{unbiased estimate}$$

- $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$
- How are $\mathrm{Cov}(X, Y)$ and $\mathrm{Cov}(Y, X)$ related?

$\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$

Estimating covariance in Matlab

– Samples

$x = [x_1 \;\; x_2 \;\; x_3 \;\; \cdots \;\; x_n]$

$y = [y_1 \;\; y_2 \;\; y_3 \;\; \cdots \;\; y_n]$

– Means

$m_x \leftarrow$ `m_x`

$m_y \leftarrow$ `m_y`

– Covariance

$$\mathrm{Cov}(X,Y) = \frac{1}{n}[x_1 - m_x \quad x_2 - m_x \quad \cdots \quad x_n - m_x] \begin{bmatrix} y_1 - m_y \\ y_2 - m_y \\ \vdots \\ y_n - m_y \end{bmatrix}$$

Method 1: `>> v = (1/n)*(x-m_x)*(y-m_y)'`

Method 2:
```
>> w = x-m_x
>> z = y-m_y
>> v = (1/n)*w*z'
```


Covariance matrix of a *D*-dimensional random vector

- In 2 dimensions

$V = \begin{bmatrix} X \\ Y \end{bmatrix}$

$$\mathrm{Cov}(\mathbf{V}) = \mathrm{E}\big((\mathbf{V}-\boldsymbol{\mu})(\mathbf{V}-\boldsymbol{\mu})^T\big)$$

$$= \mathrm{E}\left(\begin{bmatrix} X-\mu_X \\ Y-\mu_Y \end{bmatrix}\begin{bmatrix} X-\mu_X & Y-\mu_Y \end{bmatrix}\right) = \begin{bmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \mathrm{Var}(Y) \end{bmatrix}$$

- In D dimensions

$$\mathrm{Cov}(\mathbf{V}) = \mathrm{E}\big((\mathbf{V}-\boldsymbol{\mu})(\mathbf{V}-\boldsymbol{\mu})^T\big)$$

- When is a covariance matrix symmetric?

A. always, **B.** sometimes, or **C.** never

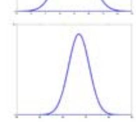
Example covariance matrix

- People's heights (made up)

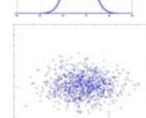
$X \sim N(67, 20)$

- Time people woke up this morning

$Y \sim N(9, 1)$



- What is the covariance matrix of $V = \begin{bmatrix} X \\ Y \end{bmatrix}$?



$\begin{bmatrix} 20 & 0 \\ 0 & 1 \end{bmatrix}$

Estimating the covariance matrix from samples (including Matlab code)

– Sample n times and find mean of samples

$$V = \begin{array}{c} \begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{array} \\ \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix} \end{array} \qquad \mathbf{m} = \begin{bmatrix} m_x \\ m_y \end{bmatrix}$$

– Find the covariance matrix

$$Cov(V) = \frac{1}{n} \begin{bmatrix} x_1 - m_x & x_2 - m_x & \cdots & x_n - m_x \\ y_1 - m_y & y_2 - m_y & \cdots & y_n - m_y \end{bmatrix} \begin{bmatrix} x_1 - m_x & y_1 - m_y \\ x_2 - m_x & y_2 - m_y \\ \vdots & \vdots \\ x_n - m_x & y_n - m_y \end{bmatrix}$$

```
>> m = (1/n)*sum(v,2)
>> z = v - repmat(m,1,n)
>> v = (1/n)*z*z'
```


Gaussian distribution in D dimensions

- 1-dimensional Gaussian is completely determined by its mean, μ, and variance, σ^2 :

$$X \sim \mathrm{N}(\mu, \sigma^2) \qquad p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- D-dimensional Gaussian is completely determined by its mean, $\boldsymbol{\mu}$, and covariance matrix, $\boldsymbol{\Sigma}$:

$$X \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma) \qquad p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2} |\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})}$$

–What happens when $D = 1$ in the Gaussian formula?