Final Exam Review Solutions

March 16, 2010

Binary Number Systems

Question 1

Show the operation of $-10 + (-5)$ in 6-bit one's complement.

First, we convert the absolute values of the operands to 6-bit binary numbers:

$$\begin{aligned} 10_{10} &= 001010_2 \\ 5_{10} &= 000101_2 \end{aligned}$$

Now we need to negate the numbers. In one's complement, we negate a number by flipping its bits:

$$\begin{aligned} -10_{10} &= 110101 \\ -5_{10} &= 111010 \end{aligned}$$

Now we add our two numbers:

$$\begin{array}{rr} & 110101 \\ + & 111010 \\ \hline & 101111 \end{array}$$

We also have a carry out of 1. We add this carry out back in to our sum:

$$\begin{array}{rr} & 101111 \\ + & 1 \\ \hline & 110000 \end{array}$$

You should verify that this is the one's complement of -15, as expected.

Question 2

Define n-digit complements for base 10. Using your definition, show the arithmetic of $-x + y$, where $x = 216_{10}$ and $y = 65_{10}$, in a 6-digit system in base 10.

The appropriate complements are nine's and ten's complements. We define both and show the arithmetic for each system.

To begin with, we'll define nine's complement in analogy to one's complement. Recall that one's complement for binary (base 2) numbers was defined as

$$-x \sim 2^n - 1 - x,$$

where n is the number of digits. So we define nine's complement as

$$-x \sim 10^n - 1 - x,$$

again where n is the number of digits. (You'll see where the "nines" bit comes in soon.)

We now convert our operands to 6-digit nine's complement:

$$\begin{array}{rcccccc} -216_{10} & \sim & 10^6 - 1 - 216 & = & 999999 - 216 & = & 999783 \\ 65_{10} & \sim & 65 & & & & \end{array}$$

And we add up the operands:

$$\begin{array}{rr} & 999783 \\ + & 65 \\ \hline & 999848 \end{array}$$

Note there is no carry out; if there were, it would have to be added back in to the sum, just as in one's complement. You should verify that the answer is -151 i nine's complement.

Now we define ten's complement in analogy to two's complement. Recall that two's complement for binary (base 2) numbers was defined as

$$-x \sim 2^n - x,$$

where n is the number of digits. So we define ten's complement as

$$-x \sim 10^n - x,$$

again where n is the number of digits. (Note that the ten's complement is just the nine's complement plus one! Note also that this is exactly the case for two's complement, which is just one's complement plus one.)

Again, we convert our operands to 6-digit ten's complement:

$$\begin{array}{rcccccc} -216_{10} & \sim & 10^6 - 216 & = & 1000000 - 216 & = & 999784 \\ 65_{10} & \sim & 65 & & & & \end{array}$$

And, once again, we add the operands:

$$\begin{array}{rr} & 999784 \\ + & 65 \\ \hline & 999849 \end{array}$$

There is no carry out this time, either, but if there were, just as in two's complement, we would not need to do anything special; it's enough to keep only the last six digits of the answer. You should verify that the answer is -151 in ten's complement.

Question 3

Define n-digit complements for base 8. Using your definition, show the arithmetic of $-x - y$, where $x = 120_8$ and $y = 27_8$, in a 6-digit system in base 8.

Pay attention! The inputs are given in *base 8*! Otherwise, this problem proceeds exactly as the previous problem did. The appropriate complements are seven's and eight's complements. We define both and show the arithmetic for each system.

First, we define seven's complement:

$$-x \sim 8^n - 1 - x,$$

where n is the number of digits. Our operands are already in base 8, so we take their complements:

$$\begin{array}{rcrcrcr} -120_8 & \sim & 8^6 - 1_8 - 120_8 & = & 777777_8 - 120_8 & = & 777657_8 \\ -27_8 & \sim & 8^6 - 1_8 - 27_8 & = & 777777_8 - 27_8 & = & 777750_8 \end{array}$$

Again, we add the operands:

$$\begin{array}{rr} & 777657 \\ + & 777750 \\ \hline & 777627 \end{array}$$

We also have a carry out, which must be added back in to the sum:

$$\begin{array}{rr} & 777627 \\ + & 1 \\ \hline & 777630 \end{array}$$

Once again, you should verify the answer.

Next, we define eight's complement:

$$-x \sim 8^n - x,$$

where n is the number of digits. Our operands are already in base 8, so we take their complements:

$$\begin{array}{rcrcrcr} -120_8 & \sim & 8^6 - 120_8 & = & 1000000_8 - 120_8 & = & 777660_8 \\ -27_8 & \sim & 8^6 - 27_8 & = & 1000000_8 - 27_8 & = & 777751_8 \end{array}$$

Again, we add the operands:

$$\begin{array}{rr} & 777660 \\ + & 777751 \\ \hline & 777631 \end{array}$$

Yet again, in this system, we do not do anything special with the carry out. Yet again, verify the answer.

Boolean Algebra

Question 1

Express Boolean function $E(x, y, z) = (x + y + x'z)'(x'y' + xy'z)$ in sum-of-products form.

$$
\begin{array}{rll}
& (x+y+x'z)'(x'y'+xy'z) & \\
= & (x+y+z)'(x'y'+xy'z) & \text{Theorem 8} \\
= & (x+y+z)'(y'(x'+xz)) & \text{Distributivity} \\
= & (x+y+z)'(y'(x'+z)) & \text{Theorem 8} \\
= & x'y'z'(y'(x'+z)) & \text{DeMorgan's Law} \\
= & x'y'z'(x'+z) & \text{Absorption} \\
= & x'y'z'x' + x'y'z'z & \text{Distributivity} \\
= & x'y'z'x' + 0 & z, z' \text{ Complements} \\
= & x'y'z' + 0 & \text{Idempotency on } x' \\
= & x'y'z' & 0 \text{ is identity for } +
\end{array}
$$

Note this is a sum of one product.

Question 2

Express Boolean function $E(x, y, z) = [(x'y + x)'(x' + y)(y' + z)]'$ in product-of-sums form.

$$
\begin{array}{rll}
& [(x'y+x)'(x'+y)(y'+z)]' & \\
= & [(y+x)'(x'+y)(y'+z)]' & \text{Theorem 8} \\
= & (y+x)'' + (x'+y)' + (y'+z)' & \text{DeMorgan's Law} \\
= & (y+x) + (x'+y)' + (y'+z)' & \text{Involution} \\
= & (y+x) + xy' + yz' & \text{DeMorgan's Law} \\
= & y + x + xy' + yz' & \text{Associativity} \\
= & y + x + yz' & \text{Absorption} \\
= & y + x & \text{Absorption}
\end{array}
$$

Note this is a product of one sum.

Question 3

Express Boolean function $E(a, b, c, d) = ab + (cd + bc)' + ad$ in sum-of-products form.

$$
\begin{array}{lr}
ab + (cd+bc)' + ad & \\
= ab + (c(d+b))' + ad & \text{Distributivity} \\
= ab + (c' + (d+b)') + ad & \text{DeMorgan's Law} \\
= ab + (c' + d'b') + ad & \text{DeMorgan's Law} \\
= ab + c' + d'b' + ad & \text{Associativity}
\end{array}
$$

You can stop here — this is sum-of-products form. Nonetheless, let's try to minimize it.

$$\begin{aligned}
&= ab + c' + d'b' + a1d && \text{Identity}\\
&= ab + c' + d'b' + a(b+b')d && \text{Complement}\\
&= ab + c' + d'b' + abd + ab'd && \text{Complement}\\
&= ab + c' + d'b' + ab'd && \text{Absorption}\\
&= ab + c' + b'(d' + ad) && \text{Absorption}\\
&= ab + c' + b'(d' + a) && \text{Absorption}\\
&= ab + c' + b'd' + b'a && \text{Absorption}\\
&= a(b+b') + c' + b'd' && \text{Absorption}\\
&= a + c' + b'd' && \text{Absorption}
\end{aligned}$$

Question 4

Express Boolean function $E(x, y, z) = [xy'(x'y + z)]'$ in product-of-sums form.

$$\begin{aligned}
& [xy'(x'y+z)]' & \\
= & [xy'x'y + xy'z)]' & \text{Distributivity}\\
= & [0 + xy'z]' & \text{Complements}\\
= & [xy'z]' & \text{0 is identity for } +\\
= & x' + y + z' & \text{DeMorgan's Law}
\end{aligned}$$

Note again that this a product of one sum.

Recursive Functions

Question 1

A frog knows 5 jumping styles, named A, B, C, D, and E. Both A and B jump forward by one foot, while C, D, and E jump forward by two feet. Let a_i be the total number of ways to jump a total distance of i feet.

1. What are a_1, a_2, and a_3?
2. Derive the recursive formulation of a_i.
3. Solve the recursion.

To start with, we note that there are only two ways to jump one foot, so

$$a_1 = 2.$$

On the other hand, we have three ways to jump two feet using only one jump. We could also jump two feet by jumping one foot twice: all of the $2^2 = 4$

combinations AA, AB, BA, and BB will work. Thus, we have the number of ways to jump two feet is

$$a_2 = 3 + 2^2 = 3 + 4 = 7.$$

Now we need to figure out how to jump three feet. Consider the very last jump: it's a jump of either one foot or two feet. So let's first consider if the last jump is one foot long. Then we have $a_2 = 7$ ways to jump the initial two feet, and we can finish off the jumps in one of two ways (A and B), so we have $2 \times a_2 = 2 \times 7 = 14$ ways to jump three feet if we finish by jumping one foot. On the other hand, the last jump may be two feet long. In this case, we have $a_1 = 2$ ways to jump the first foot, and we can finish off the jumps in one of three ways (C, D, and E), so we have $3 \times a_1 = 3 \times 2 = 6$ ways to jump three feet if we finish by jumping two feet. Finally, we add all the ways to finish, both by making a final jump of two feet and one foot, to get

$$a_3 = 14 + 6 = 20.$$

To figure out the recusion, we just generalize the argument we used for a_3: if we want to jump i feet, we can either jump $i - 1$ feet and finish with A or B, or we can jump $i - 2$ feet and finish with C, D, or E. Thus, we have

$$a_i = 2a_{i-1} + 3a_{i-2}.$$

Finally, we have to solve the recurrence. The characteristic polynomial of the recurrence relation above is

$$x^2 - 2x - 3 = (x - 3)(x + 1).$$

This polynomial has the roots

$$\begin{aligned} r_1 &= 3 \\ r_2 &= -1 \end{aligned}$$

The general form of the solution to this recurrence is thus

$$a_i = c_1 r_1^i + c_2 r_2^i = 3^i c_1 + (-1)^i c_2.$$

We have only to solve for c_1 and c_2 using our initial conditions. We have the two equations

$$\begin{array}{lllllll} a_1 & = & 3^1 c_1 + (-1)^1 c_2 & = & 3c_1 + (-1)c_2 & = & 2 \\ a_2 & = & 3^2 c_1 + (-1)^2 c_2 & = & 9c_1 + c_2 & = & 7 \end{array}$$

Adding these equations together, we get

$$a_1 + a_2 = 12c_1 = 9$$

and therefore

$$c_1 = \frac{3}{4}.$$

Plugging this into the equation for a_1 yields

$$c_2 = \frac{1}{4}.$$

Thus, the solution to the recurrence is

$$a_i = 3^i \frac{3}{4} + (-1)^i \frac{1}{4} = \frac{1}{4}(3^{i+1} + (-1)^i).$$

Question 2

Find the solution of the following recurrence:

$$\begin{array}{rcl} a_n & = & -a_{n-1} + a_{n-2} + a_{n-3} \\ a_0 & = & 0 \\ a_1 & = & 0 \\ a_2 & = & 1 \end{array}$$

We first find the characteristic polynomial of the recurrence relation, which is

$$x^3 + x^2 - x - 1 = (x-1)(x+1)^2.$$

This polynomial has the roots

$$\begin{array}{rcl} r_1 & = & 1 \\ r_2 = r_3 & = & -1 \end{array}$$

Note that we have a root with multiplicity two (r_2 and r_3). Thus, the general form of the solution will be

$$a_n = c_1 r_1^n + c_2 r_2^n + c_3 n r_3^n = c_1 + (-1)^n c_2 + (-1)^n n c_3.$$

Again, we solve for the constants using our initial conditions. We have the equations:

$$\begin{array}{rclclcl} a_0 & = & c_1 + (-1)^0 c_2 + (-1)^0 0 c_3 & = & c_1 + c_2 & = & 0 \\ a_1 & = & c_1 + (-1)^1 c_2 + (-1)^1 1 c_3 & = & c_1 + (-1)c_2 + (-1)c_3 & = & 0 \\ a_2 & = & c_1 + (-1)^2 c_2 + (-1)^2 2 c_3 & = & c_1 + c_2 + 2c_3 & = & 1 \end{array}$$

Solving this system of linear equations yields the constants

$$\begin{array}{rcl} c_1 & = & \frac{1}{4} \\ c_2 & = & -\frac{1}{4} \\ c_3 & = & \frac{1}{2} \end{array}$$

Thus, the solution to the recurrence is

$$a_n = \frac{1}{4} + (-1)^n (\frac{1}{2} n - \frac{1}{4}).$$

Question 3

Consider the homogeneous linear recurrence relation

$$a_n = 3ra_{n-1} - 3r^2 a_{n-2} + r^3 a_{n-3}.$$

Show that

$$p(n) = c_1 r^n + c_2 n r^n + c_3 n^2 r^n = r^n(c_1 + c_2 n + c_3 n^2)$$

satisfies the recurrence relation, where c_1, c_2, and c_3 are constant coefficients.

We have to show that, if we plug in $p(n)$ in place of each a_n on the right hand side of the recurrence, we get $p(n)$ back out.

$$\begin{aligned}
& 3rp(n-1) - 3r^2 p(n-2) + r^3 p(n-3) \\
= \; & 3r(r^{n-1}(c_1 + c_2(n-1) + c_3(n-1)^2)) \\
& -3r^2(r^{n-2}(c_1 + c_2(n-2) + c_3(n-2)^2)) \\
& +r^3(r^{n-3}(c_1 + c_2(n-3) + c_3(n-3)^2)) \\
= \; & 3r^n(c_1 + c_2(n-1) + c_3(n-1)^2) \\
& -3r^n(c_1 + c_2(n-2) + c_3(n-2)^2) \\
& +r^n(c_1 + c_2(n-3) + c_3(n-3)^2) \\
= \; & r^n[3(c_1 + c_2(n-1) + c_3(n-1)^2) \\
& -3(c_1 + c_2(n-2) + c_3(n-2)^2) \\
& +(c_1 + c_2(n-3) + c_3(n-3)^2)] \\
= \; & r^n[3c_1 + (3n-3)c_2 + (3n^2 - 6n + 3)c_3 \\
& -3c_1 - (3n-6)c_2 - (3n^2 - 12n + 12)c_3 \\
& +c_1 + (n-3)c_2 + (n^2 - 6n + 9)c_3] \\
= \; & r^n(c_1 + nc_2 + n^2 c_3) \\
= \; & p(n)
\end{aligned}$$

Pigeonhole Principle

4.1 Solution:

First, notice that by dividing a circle into six 60° sectors, in each sector all points have distance $\leq$ radius. But we have only 6 points, arbitrarily dividing the circle leads to 6 pigeons in 6 holes.

We can prove it by a more careful dividing.

Proof:

Pick any point P among the 6 points. If P is on the center O of the circle, then the distance from P to any other point doesn't exceed the radius; otherwise, draw a diameter on P, and draw two diameters 60° to the first diameter, so that the circle is divided into six 60° sectors.

Now, if there is another point Q in one of the two sectors on P, then $|PQ| \leq$ radius; otherwise, all the 5 remaining points are in the other 4 sectors, and by Pigeonhole principle at least 2 points in one sector, with distance $\leq$ radius.

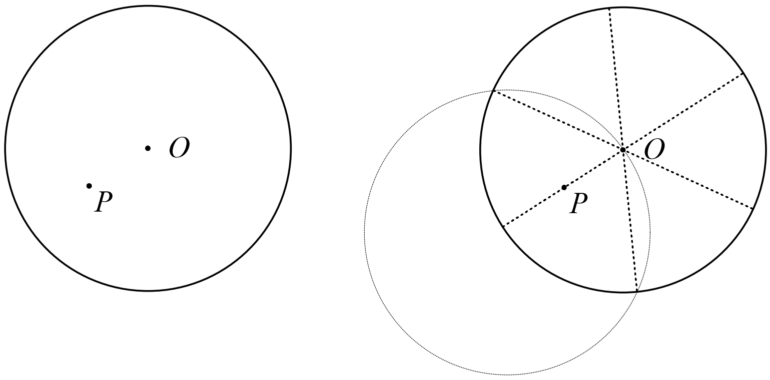

Figure 1: Pick any point P, and divide the circle by P

4.2 Proof: (by contradiction)

First, since p is prime and $a < p$, p is not a's divisor. Also for any $b_1 \neq b_2$ in range $0 < b_1, b_2 < p$, p is not $(b_1 - b_2)$'s divisor. So $(b_1 - b_2)a\%p \neq 0$, and therefore $b_1 a\%p \neq b_2 a\%p$.

We look at all the values of $(p-1)$ residuals from $(1a\%p)$, $(2a\%p)$, ..., to $(p-1)a\%p$. If none of these are equal to 1, then the possible values of these residuals are $2, 3, 4, \cdots, p-1$, totally $(p-2)$ values. By Pigeonhole principle, there must be two residuals having the same value, which means $b_1 \neq b_2$ with $b_1 a\%p = b_2 a\%p$, contradicting the previous statement we prove. So the assumption that "none of these are equal to 1" must be false, theorem proved.

4.3 Proof:

Put the row of 12 chairs into 4 groups, with the ith group containing the $(3i-2)$th, $(3i-1)$th and $(3i)$th chair (which are consecutive). With 9 people seated in these chairs, by generalized pigeonhole principle, there must be a group containing 3 people, so the 3 consecutive seats in this group have people in them.

Counting

5.1 Solution:

We can count the zeros by their positions.

Lowest digit: one zero every 10 numbers, so we have $1000 \div 10 = 100$ zeros from 1 to 1000.

Second lowest digit: from 100, 101, ..., 109 to 900, 901, ..., 909, and finally 1000 also has one. Totally $10 \times 9 + 1 = 91$.

Third digit, only 1000 has a zero on the third digit.

Answer: 100 + 91 + 1 = 192

5.2 Solution:

To have the shortest walking distance, each route has exactly 8 horizontal segments and 10 vertical segments. Any order of these segments combined is a way of walking from (0,0) to (8,10), so the number of ways is $C(10+8,8) = 43758$.

Alternative solution:

On the grid from (0,0) to (8,10), let x_i ($0 \le i \le 8$) denote the distance to walk towards north on the $(i+1)$th vertical line. Then

$x_i \ge 0$, (no detour allowed);

$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 10$ (destination on (8,10));

So the number of ways is the number of solutions to the equation, which is $C(10+9-1, 9-1) = 43758$.

5.3 Solution:

We can calculate all the sets using the standard procedure (like in the next problem). But here by observation, we can find that when w, y, z all taking their maximum values, $w + y + z = 7 + 3 + 9 = 19 < 29$. So for any set of valid (w, y, z), $x = 29 - w - y - z$ is always greater than one, i.e. (w, x, y, z) is always a valid solution to the equation. So the number of nonnegative solutions is exactly the number of combinations on picking (w, y, z), which is $8 \times 4 \times 10 = 320$.

5.4 Solution:

Let $z' = 4 - z$, so $0 \le z' \le 4$, and the equation becomes $x + y + z' = 19$. The total number of nonnegative solutions (without constraints) is $C(19+2,2) = 210$.

Let A denote the set of solutions with $x \ge 8$;

Let B denote the set of solutions with $y \ge 9$;

Let C denote the set of solutions with $z' \ge 5$;

$|A| = C(19-8+2,2) = 78$

$|B| = C(19-9+2,2) = 66$

$|C| = C(19-5+2,2) = 120$

$|A \cap B| = C(19-8-9+2,2) = 6$

$|A \cap C| = C(19-8-5+2,2) = 28$

$|B \cap C| = C(19-9-5+2,2) = 21$

$|A \cap B \cap C| = C(19-8-9-5+2,2) = 0$

So $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 209$

Answer $= C(19+2,2) - |A \cup B \cup C| = 1$.

Alternative solution:

Observe that if we pick x's and y's maximum values, $x + y = 7 + 8 = 15$. Only in this case we can pick $z = 0$ which is a nonnegative solution. Any other (x, y) pair will lead to $z < 0$ and not counted as a valid solution. So the number of solution(s) is only 1.

5.4 Proof:

We already have the 2-set theorem $|A \cup B| = |A| + |B| - |A \cap B|$.

$|A \cup B \cup C|$
$= |(A \cup B) \cup C|$
$= |A \cup B| + |C| - |(A \cup B) \cap C|$ // 2-set theorem
$= |A \cup B| + |C| - |(A \cap C) \cup (B \cap C)|$ // distributivity
$= |A| + |B| - |A \cap B| + |C| - (|A \cap C| + |B \cap C| - |A \cup B \cup C|)$ // 2-set
$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cup B \cup C|$

5.5 Proof:

$|A \cup B \cup C \cup D|$
$= |(A \cup B \cup C) \cup D|$
$= |A \cup B \cup C| + |D| - |(A \cup B \cup C) \cap D|$ // 2-set theorem
$= |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cup B\cup C|+|D|-|(A\cup B\cup C)\cap D|$
// 3-set theorem
$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| + |D|$
$-|(A \cap D) \cup (B \cap D) \cup (C \cap D)|$ // distributivity
$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| + |D|$
$-(|A \cap D| + |B \cap D| + |C \cap D| - |A \cap B \cap D| - |A \cap C \cap D|$
$-|B \cap C \cap D| + |A \cap B \cap C \cap D|)$ // 3-set theorem
$= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D|$
$-|C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$
$-|A \cap B \cap C \cap D|$