1 Binary Number Systems

1. (one's complement) Show the operation of $-10+(-5)$ in 6-bit one's complement.
2. (two's and one's complements) We have defined and learned the idea of two's and one's complements for n-bit binary numbers. Define the corresponding complements using an n-digit system with base 10. Show the arithmetic of $-x+y$ where $x = 216_{10}$ and $y = 65_{10}$ in the corresponding complement representations using a 6-digit system with base 10.
3. (two's and one's complements) We have defined and learned the idea of two's and one's complements for n-bit binary numbers. Define the corresponding complements using an n-digit system with base 8. Show the arithmetic of $-x-y$ where $x = 120_8$ and $y = 27_8$ in the corresponding complement representations using a 6-digit system with base 8.

2 Boolean Algebra

1. (expression in sum of products) Express Boolean function
$E(x, y, z) = (x + y + x'z)'(x'y' + xy'z)$ in sum-of-products form.
2. (expression in product of sums) Express Boolean function
$E(x, y, z) = [(x'y + x)'(x' + y)(y' + z)]'$ in product-of-sums form.
3. (expression in sum of products) Express Boolean function
$E(a, b, c, d) = ab + (cd + bc)' + ad$ in sum-of-products form.
4. (expression in product of sums) Express Boolean function
$E(x, y, z) = [xy'(x'y + z)]'$ in product-of-sums form.

3 Recursive function

1. A frog knows 5 jumping styles (A, B, C, D, E). A, B jump forward by 1 foot, and C, D, E jump forward by 2 feet. Let a_i denote the number of ways to jump over a total distance of i feet.
(a) What is a_1, a_2, a_3?
(b) Derive the recursive formula of a_n?
(c) Find the solution of the recursion.
2. Find the solution of the following recurrence:

$$\begin{aligned}
a_n &= -a_{n-1} + a_{n-2} + a_{n-3} \\
a_0 &= 0 \\
a_1 &= 0 \\
a_2 &= 1
\end{aligned}$$

3. Consider the following homogeneous linear recurrence relation:
$a_n = 3ra_{n-1} - 3r^2 a_{n-2} + r^3 a_{n-3}$. Show that $a_n = c_1 r^n + c_2 n r^n + c_3 n^2 r^n$ satisfies the recurrence relation, where c_1, c_2, and c_3 are constant coefficients.

4 Pigeonhole principle

1. (points in a circle area) Put 6 points in a plane circle area, prove there are 2 points with distance $\leq$ radius.
2. (reverse of multiplication) Assume that p is a prime number. Prove that for any non-zero integer a with $0 < a < p$, there is an integer $0 < b < p$ such that $(ab)\%p = 1$.
3. (seating in a row) 9 people are seated in a row of 12 chairs. Prove that there must be at least three consecutive seats with people in them.

5 Counting

1. (counting numbers) How many zeros do we need to write from 1 to 1000? (For example, we need one zero for each in a set {10, 20, 90, 109, 906}, two zeros in a set {100, 300, 900}.)
2. (possible routing paths) How many ways to walk from $(0, 0)$ to $(8, 10)$, assuming the streets are all on a grid, and the walking distance must be shortest.
3. (integer linear equation) Find the number of nonnegative integer solutions to $w + x + y + z = 29$ with constraints that $w < 8, x > 1, y < 4, z < 10$.
4. (integer linear equation) Find the number of nonnegative integer solutions to $x + y - z = 15$ with constraints that $x < 8, y < 9, z < 5$.
5. (inclusion and exlusion theorem) Prove the inclusion and exclusion theorem when the number of sets is 3, as stated in the following equation.
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C|$
$- |B \cap C| + |A \cap B \cap C|$, where $|X|$ is the number of elements in set X.
6. (inclusion and exlusion theorem) Prove the inclusion and exclusion theorem when the number of sets is 4, as stated in the following equation.
$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C|$
$- |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$