Question 1 (10 Points)

Show the operation of 11×23 in a residual number system with moduli $(m_1, m_2, m_3) = (7, 8, 9)$.

Solution

First, we convert 11 to the given residual number system:

$$(11\%7, 11\%8, 11\%9) = (4, 3, 2)$$

Next, we convert 23:

$$(23\%7, 23\%8, 23\%9) = (2, 7, 5)$$

We multiply the numbers pairwise and take the mod of the result:

$$(4 \times 2\%7, 3 \times 7\%8, 2 \times 5\%9) = (1, 5, 1)$$

Grading Policy

If you only converted the answer without showing any work, you lost 7 points. If you made an error in conversion, you lost 2 points. If you forgot to mod after multiplying, you lost 3 points.

Question 2 (15 Points)

Suppose $(x\%5, x\%6, x\%7) = (1, 3, 5)$, where symbol % denotes modulus operation. Find the smallest positive integer x that satisfies this system.

Solution & Grading Policy

$M = m_1 m_2 m_3 = 5 \times 6 \times 7 = 210$	3pts
$M_1 = 6 \times 7 = 42$	1pts
$M_1 s_1 \% m_1 = r_1 \Rightarrow s_1 = 3$	2pts
$M_2 = 5 \times 7 = 35$	1pts
$M_2 s_2 \% m_2 = r_2 \Rightarrow s_2 = 5$	2pts
$M_1 = 5 \times 6 = 30$	1pts
$M_1 s_3 \% m_3 = r_3 \Rightarrow s_3 = 4$	2pts
$x = (M_1 s_1 r_1 + M_2 s_2 r_2 + M_3 s_3 r_3)\%M = 201$	3pts

Question 3 (15 Points)

Express Boolean function

$$E(x, y, z) = (x + y + z)(x'y' + xy'z)'$$

in sum-of-products form.

Solution

$$
\begin{aligned}
(x+y+z)(x'y'+xy'z)' &= (x+y+z)(x'y')'(xy'z)' && \text{DeMorgan's Law}\\
&= (x+y+z)(x+y)(x'+y+z') && \text{DeMorgan's Law}\\
&= (x+y)(x'+y+z') && \text{Absorption}\\
&= (y+x)(y+x'+z') && \text{Commutativity}\\
&= y+x(x'+z') && \text{Distributivity}\\
&= y+xx'+xz' && \text{Distributivity}\\
&= y+0+xz' && \text{Complements}\\
&= y+xz' && \text{Identity}
\end{aligned}
$$

Grading Policy

If you successfully applied DeMorgan's law to remove the complement, you got 2 points. If your answer was a normal form (sum-of-products or product-of-sums), you got 2 points. If your answer was in sum-of-products form, you got 3 points. A correct derivation was worth 8 points; generally a point was taken off per error, though more points could be taken for more serious errors.

Question 4 (20 Points)

Express Boolean function

$$E(x,y,z) = x'y + x[(x'+y)(y'+z)]'$$

in sum-of-products form.

Solution

$$
\begin{aligned}
x'y + x[(x'+y)(y'+z)]' &= x'y + x[(x'+y)' + (y'+z)'] && \text{DeMorgan's Law}\\
&= x'y + x[xy' + yz'] && \text{DeMorgan's Law}\\
&= x'y + xy' + xyz' && \text{Distributivity}\\
&= (x'+xy'+xyz')(y+xy'+xyz') && \text{Distributivity}\\
&= (x'+y'+yz')(y+x+xyz') && \text{Theorem8}\\
&= (x'+y'+z')(y+x+xyz') && \text{Theorem8}\\
&= (x'+y'+z')(y+x) && \text{Absorption}
\end{aligned}
$$

Grading Policy

If you correctly applied DeMorgan's law to remove the complement out of parentheses, 10 points. A correct sum-of-product form, 15 points. A correct product-of-sum form, all 20 points. Points are deducted for any errors.

Question 5 (20 Points)

Prove or disprove that, for any elements a, b, and c in set B of a Boolean algebra,

$$(a' + c)(a + b)(b + c) = (a' + c)(a + b)$$

.

Solution

We work the left and right hand sides separately. We begin with the left side:

$$\begin{aligned}
(a'+c)(a+b)(b+c) &= (a'+c)(b+a)(b+c) && \text{Commutativity} \\
&= (a'+c)(b+ac) && \text{Distributivity} \\
&= a'b + a'ac + cb + cac && \text{Distributivity} \\
&= a'b + 0c + cb + acc && \text{Distributivity, Commutativity} \\
&= a'b + 0 + cb + ac && \text{Boundedness, Idempotency} \\
&= a'b + cb + ac && \text{Identity}
\end{aligned}$$

Now the right side:

$$\begin{aligned}
(a'+c)(a+b) &= a'a + a'b + ca + cb && \text{Distributivity} \\
&= 0 + a'b + ca + cb && \text{Complements} \\
&= a'b + ca + cb && \text{Identity} \\
&= a'b + cb + ca && \text{Commutativity} \\
&= a'b + cb + ac && \text{Commutativity}
\end{aligned}$$

Since we reduced both the left and right hand sides to the same expression, the two initial expressions are equal.

Grading Policy

If you said the expressions were not equal or did not answer, you lost 10 points. If your derivation included at least 5 steps, you got different amounts of bonus "dilligence" points: 5 points if you did not claim the expressions were not equal, and 3 points if you did. If your derivation included fewer than 3 steps, you lost 5 points. For each $\frac{1}{4}$ of your derivation which was wrong, you lost 5 points. These are general guidelines that applied to the majority of exams; some special cases that didn't fall into these categories received different amounts of points.

Question 6 (20 Points)

Reduce the following to an expression of a minimal number of literals (3): $E(a, b, c) = abc + ac'd + bc'd' + a'b'c' + ab'c'd' + bc'd$

Solution

$$\begin{array}{lll}
& abc + ac'd + bc'd' + a'b'c' + ab'c'd' + bc'd & \\
= & abc + ac'd + a'b'c' + ab'c'd' + bc'(d' + d) & \text{Commutativity + Distributivity} \\
= & abc + ac'd + b'c'(a' + ad') + bc' & \text{Distributivity} \\
= & abc + ac'd + b'c'(a' + d') + bc' & \text{Theorem8} \\
= & abc + ac'd + a'b'c' + b'c'd' + bc' & \text{Distributivity} \\
= & abc + c'(ad + a'b' + b'd' + b) & \text{Distributivity} \\
= & abc + c'(ad + a' + d' + b) & \text{Theorem8} \\
= & abc + c'(a + a' + d' + b) & \text{Theorem8} \\
= & abc + c'(1 + d' + b) & \text{Identity} \\
= & abc + c' & \text{Identity} \\
= & ab + c' & \text{Theorem8}
\end{array}$$

Grading Policy

$ab + c'$ is the only result to have full credit. You lose points if your derivations are wrong or not shortening the statement. 16 points are for reducing the 19-literal statement to the 3-literal one, where we look for the shortest correct line among the derivations, and give (19 - #literals) points. The remaining 4 points are for the laws/theorems applied in derivations.