Solution of CSE20 Final Exam

March 25, 2010

Question 1 (7.5 Points)

Show the operation of $-19+(-4)$ in a one's complement binary number system. Assume that each binary number is represented by 10 bits.

Solution

We first convert each number to its 10-bit one's complement. We begin by converting the magnitude of each number to a 10-bit binary number:

$$\begin{aligned} 19_{10} &= 0000010011_2 \\ 4_{10} &= 0000000100_2 \end{aligned}$$

We then flip the bits to get the representation of the negative values:

$$\begin{aligned} -19_{10} &= 1111101100 \\ -4_{10} &= 1111111011 \end{aligned}$$

We then add the two values:

$$\begin{array}{rr} & 1111101100 \\ + & 1111111011 \\ \hline & 1111100111 \end{array}$$

There is a carry out of one, which has to be added back in to the sum:

$$\begin{array}{rr} & 1111100111 \\ + & 1 \\ \hline & 1111101000 \end{array}$$

This is the final answer. We can verify the magnitude is correct by flipping the bits again, which yields 10111_2, which is 23_{10}, as expected.

Grading Policy

.5 points were deducted for each conversion error. Small arithmetic errors (for example, dropping a carry) lost 1 point; large arithmetic errors (usually ones that made the answer no longer negative) lost 2 points. Forgetting the end-around carry lost one point. Using fewer than 10 bits lost 4 points. Adding an extra bit lost 1 point. Performing arithmetic with positive numbers lost 4 points. Getting everything right but forgetting to write the final answer lost 1 point.

Question 2 (7.5 Points)

We have defined and learned the idea of two's and one's complements for n-bit binary numbers. Define the corresponding complements using an n-digit system with base 10. Show the arithmetic of $-x-y$ where $x = 123_{10}$ and $y = 45_{10}$ in the corresponding complement representations using a 5-digit system with base 10.

Solution

The appropriate complements are nine's and ten's complements. We define both and show the arithmetic for each system.

To begin with, we'll define nine's complement in analogy to one's complement. Recall that one's complement for binary (base 2) numbers was defined as

$$-x \sim 2^n - 1 - x,$$

where n is the number of digits. So we define nine's complement as

$$-x \sim 10^n - 1 - x,$$

again where n is the number of digits. (You'll see where the "nines" bit comes in soon.)

We now convert our operands to 5-digit nine's complement:

$$\begin{array}{rcrcrcl} -123_{10} & \sim & 10^5 - 1 - 123 & = & 99999 - 123 & = & 999876 \\ -45_{10} & \sim & 10^5 - 1 - 45 & = & 99999 - 45 & = & 999954 \end{array}$$

And we add up the operands:

$$\begin{array}{rr} & 999876 \\ + & 999954 \\ \hline & 999830 \end{array}$$

There is a carry out which must be added back in to the sum:

$$\begin{array}{rr} & 999830 \\ + & 1 \\ \hline & 999831 \end{array}$$

This is the final answer for nine's complement.

Now we define ten's complement in analogy to two's complement. Recall that two's complement for binary (base 2) numbers was defined as

$$-x \sim 2^n - x,$$

where n is the number of digits. So we define ten's complement as

$$-x \sim 10^n - x,$$

again where n is the number of digits. (Note that the ten's complement is just the nine's complement plus one! Note also that this is exactly the case for two's complement, which is just one's complement plus one.)

Again, we convert our operands to 5-digit ten's complement:

$$\begin{array}{rcrcrcr} -123_{10} & \sim & 10^5 - 123 & = & 100000 - 123 & = & 999877 \\ -45_{10} & \sim & 10^5 - 45 & = & 100000 - 45 & = & 999955 \end{array}$$

And, once again, we add the operands:

$$\begin{array}{rr} & 999877 \\ + & 999955 \\ \hline & 999832 \end{array}$$

This is the final answer for ten's complement.

Grading Policy

Typo-type errors lost 1 point. Showing the answer in only one system lost 2 points (it was ok if at least the conversions were shown in both systems). A missing or unnecessary end-around carry lost 1 point. Subtracting instead of adding 1 for ten's complement lost 2 points. Subtracting 45 without converting it to its complement lost 2 points. Incorrect arithmetic lost 3 points. Doing the conversions without showing any arithmetic lost 3 points.

Question 3 (7.5 Points)

Express Boolean function
$E(x, y, z) = (xy + x'z)'(xz' + y'z)$ in sum-of-products form.

Solution

$$
\begin{array}{lll}
 & (xy + x'z)'(xz' + y'z) & \\
= & (xy)'(x'z)'(xz' + y'z) & \text{DeMorgan's Law} \\
= & (x' + y')(x + z')(xz' + y'z) & \text{DeMorgan's Law} \\
= & (x'x + x'z' + y'x + y'z')(xz' + y'z) & \text{Distributivity} \\
= & (0 + x'z' + y'x + y'z')(xz' + y'z) & \text{Complements} \\
= & (x'z' + y'x + y'z')(xz' + y'z) & \text{Identity} \\
= & x'z'xz' + x'z'y'z + y'xxz' + y'xy'z + y'z'xz' + y'z'y'z & \text{Distributivity} \\
= & 0 + 0 + y'xxz' + y'xy'z + y'z'xz' + 0 & \text{Complements} \\
= & y'xxz' + y'xy'z + y'z'xz' & \text{Identity} \\
= & y'xz' + y'xz + y'xz' & \text{Idempotency} \\
= & y'xz + y'xz' & \text{Idempotency} \\
= & y'x(z + z') & \text{Distributivity} \\
= & y'x1 & \text{Complements} \\
= & y'x &
\end{array}
$$

Grading Policy

You earned 1.5 points for deriving a normal form (either sum-of-products or product-of-sums). You earned 2 more points if it was the correct normal form. You earned 2 points for applying DeMorgan's law correctly. You earned another 2 points if your derivation was correct; each error in a derivation lost a point, with no limit to how many points could be lost.

Question 4 (7.5 Points)

Express Boolean function
$E(x, y, z) = x'yz + [(x' + y)(y' + z)]'$ in product-of-sums form.

Solution

$$
\begin{array}{rll}
& x'yz + [(x'+y)(y'+z)]' & \\
= & x'yz + [(x'+y)' + (y'+z)'] & \text{DeMorgan's Law} \\
= & x'yz + [xy' + yz'] & \text{DeMorgan's Law} \\
= & x'yz + xy' + yz' & \text{Associativity} \\
= & x'yz + yz' + xy' & \text{Commutativity} \\
= & (x'yz + yz' + x)(x'yz + yz' + y') & \text{Distributivity} \\
= & (yz + yz' + x)(x'yz + yz' + y') & \text{Theorem 8} \\
= & (y(z+z') + x)(x'yz + yz' + y') & \text{Distributivity} \\
= & (y1 + x)(x'yz + yz' + y') & \text{Complements} \\
= & (y + x)(x'yz + yz' + y') & \text{Complements} \\
= & (y + x)(x'yz + z' + y') & \text{Theorem 8} \\
= & (y + x)(x'z + z' + y') & \text{Theorem 8} \\
= & (y + x)(x' + z' + y') & \text{Theorem 8}
\end{array}
$$

Grading Policy

Same as the previous question.

Question 5 (10 Points)

A frog knows 3 jumping styles (A, B, C). With style A the frog jumps forward by 1 foot, and with styles B, C, the frog jumps forward by 2 feet. Let a_i denote the number of ways to jump over a total distance of i feet.
(a) What is a_1, a_2, a_3?
(b) Derive the recursive formula of a_n.
(c) Find the solution of the recursion.

Solution

To start with, we note that there's only one way to jump one foot, so

$$a_1 = 1.$$

On the other hand, we have two ways to jump two feet using only one jump. We could also jump two feet by jumping one foot twice. Thus, we have the number of ways to jump two feet is

$$a_2 = 2 + 1 = 3.$$

Now we need to figure out how to jump three feet. Consider the very last jump: it's a jump of either one foot or two feet. So let's first consider if the last jump is one foot long. Then we have $a_2 = 3$ ways to jump the initial two feet, and we can only finish off the jumps in one way (A). On the other hand, the last jump may be two feet long. In this case, we have $a_1 = 1$ way to jump the first foot, and we can finish off the jumps in one of two ways (B and C), so we have

$2 \times a_1 = 1 \times 2 = 2$ ways to jump three feet if we finish by jumping two feet. Finally, we add all the ways to finish, both by making a final jump of two feet and one foot, to get

$$a_3 = 2 + 3 = 5.$$

To figure out the recusion, we just generalize the argument we used for a_3: if we want to jump i feet, we can either jump $i-1$ feet and finish with A, or we can jump $i-2$ feet and finish with B or C. Thus, we have

$$a_i = a_{i-1} + 2a_{i-2}.$$

Finally, we have to solve the recurrence. The characteristic polynomial of the recurrence relation above is

$$x^2 - x - 2 = (x-2)(x+1).$$

This polynomial has the roots

$$\begin{aligned} r_1 &= 2 \\ r_2 &= -1 \end{aligned}$$

The general form of the solution to this recurrence is thus

$$a_i = c_1 r_1^i + c_2 r_2^i = 2^i c_1 + (-1)^i c_2.$$

We have only to solve for c_1 and c_2 using our initial conditions. We have the two equations

$$\begin{aligned} a_1 &= 2^1 c_1 + (-1)^1 c_2 &= 2c_1 + (-1)c_2 &= 1 \\ a_2 &= 2^2 c_1 + (-1)^2 c_2 &= 4c_1 + c_2 &= 3 \end{aligned}$$

Adding these equations together, we get

$$a_1 + a_2 = 6c_1 = 4$$

and therefore

$$c_1 = \frac{2}{3}.$$

Plugging this into the equation for a_1 yields

$$c_2 = \frac{1}{3}.$$

Thus, the solution to the recurrence is

$$a_i = 2^i \frac{2}{3} + (-1)^i \frac{1}{3} = \frac{1}{3}(2^{i+1} + (-1)^i).$$

Grading Policy

There were 3 parts to the question

Part (a) was worth 2 points. 1 point was deducted per error.

Part (b) was worth 3 points. If your formula contained no recursive instances (i.e., no a_i), you lost 3 points. Formulas which somehow accounted for previous values in the series but were got 2 points. Formulas which seemed to be arbitrary got 1 point.

Part (c) was worth 5 points. You got 2 points for finding the correct roots of the characteristic polynomial. You got another 2 points for a correct closed formula. You got a final point for correctly solving for the constants.

Question 6 (10 Points)

Consider the following homogeneous linear recurrence relation: $a_n = 4a_{n-1} + 5a_{n-2} + 2a_{n-3}$. Show that $an = c_1 2^n + c_2 n + c3$ satisfies the recurrence relation, where c_1, c_2, and c_3 are constant coefficients.

Solution

Plug $a_n = c_1 2^n + c_2 n + c3$ into the recurrence relation.

L.H.S. $= a_n = c_1 2^n + c_2 n + c3$

R.H.S. $= 4a_{n-1} + 5a_{n-2} + 2a_{n-3}$

$$
\begin{aligned}
&= 4[c_1 2^{n-1} + c_2(n-1) + c3] - 5[c_1 2^{n-2} + c_2(n-2) + c3] \\
&\quad +2[c_1 2^{n-3} + c_2(n-3) + c3] \\
&= c_1[4 \times 2^{n-1} - 5 \times 2^{n-2} + 2 \times 2^{n-3}] + c_2[4(n-1) - 5(n-2) + 2(n-3)] \\
&\quad +c_3[4 - 5 + 2] \\
&= c_1 2^n + c_2 n + c3
\end{aligned}
$$

So L.H.S. = R.H.S., proof complete.

Grading Policy

3 points for correctly plugging in the a_n formula into the recurrence relation, and 7 points for correct deductions (-1 each error). If you show only an example like when $n = 4$, L.H.S. $= a_4 = c_1 2^4 \cdots$, you get half of points.

Question 7 (10 Points)

A team plays 30 games in a 20-day period, and plays at least one game a day. Show that there is a period of days in which exactly 10 games were played.

Solution

Define number sequence $(s_0, \cdots, s_{20})$ where s_i is the number of games played in the early i days (from day 1 to day i). Then define sequence $(t_0, \cdots, t_{20})$ where $t_i = s_i + 10$. The two sequences have totally 42 numbers.

Since 30 games are played in these days, we have $s_{20} = 30$ and $t_{20} = s_{20} + 10 = 40$. So the range of all the numbers in sequence (s_i) and (t_i) is [0, 40], which has 41 values.

By pigeonhole principle, 42 numbers in a range of 41 values, at least two numbers have the same value. And since $s_i < s_{i+1}$, $t_i < t_{i+1}$ (because they play at least one game a day), there must be some (i, j) such that $s_i = t_j$. So $s_i = s_j + 10$, $s_i - s_j = 10$, which means 10 games are played from day $(j + 1)$ to day i.

Grading Policy

There are other ways to prove the statement, however they are usually more complicated, and no one in this class actually got the complete proof without using pigeonholes. For the pigeonhole principle, setting up the two sequences s_i and t_i gets 4 points, number of pigeons 3 points and number of holes 3 points.

Question 8 (10 Points)

Ten different integer numbers are chosen from an interval [11, 30]. Show that at least three pairs of numbers have the same difference.

Solution

Count of number pairs among 10 numbers is $C(10, 2) = 45$.

The maximum difference is $30 - 11 = 19$ and the minimum difference is 1, so the range of all pairs' difference has 19 values.

$45 > 19 * 2 = 38$. By general pigeonhole principle, at least $(2 + 1 =)$ 3 pairs have the same difference.

Grading Policy

2 points for $C(10, 2)$; 2 points for the range [1, 19] ([0, 19] also works here); 3 points for #pigeons and 3 points for #holes.

Question 9 (10 Points)

How many ways to walk from (0, 0) to (5, 8), assuming that the streets are all on a grid, and the walking distance must be shortest.

Solution

To have the shortest walking distance, each route has exactly 5 horizontal segments and 8 vertical segments. Any order of these segments combined is a way of walking from (0,0) to (5,8), so the number of ways is $C(5 + 8, 5) = 1287$.

OR

On the grid from (0,0) to (5,8), let x_i ($0 \le i \le 5$) denote the distance to walk towards north on the $(i+1)$th vertical line. Then

$x_i \ge 0$, (no detour allowed);

$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 8$ (destination on (5,8));

So the number of ways is the number of solutions to the equation, which is $C(8+6-1, 6-1) = 1287$.

Grading Policy

If you map the ways of walking to the solutions of a equation correctly, you get 4 points, then 3 points for $C(5+8,5)$; otherwise 7 points for $C(5+8,5)$. Remaining 3 points for correct result.

Question 10 (10 Points)

Find the number of nonnegative integer solutions to $x+y+z = 9$ with constraints that $x > 1, y < 5, z < 6$.

Solution

By the constraint $x > 1$, i.e. $x \ge 2$, we take $x' = x - 2$, and the equivalent problem is to find the number of nonnegative integer solutions to $x' + y + z = 7$ with constraints $y < 5, z < 6$.

Let U denote the solutions with no constraint, $|U| = C(7+2,2) = 36$;

Let A denote the set of solutions with $x \ge 5$, $|A| = C(7-5+2,2) = 6$;

Let B denote the set of solutions with $y \ge 6$, $|B| = C(7-6+2,2) = 3$;

$|A \cap B| = C(7-5-6+2,2) = 0$;

$|U - A \cup B| = |U| - |A| - |B| + |A \cap B| =$ 36 - 6 - 3 + 0 = 27

The number of solutions is 27.

OR

Observe that under constraints $y < 5, z < 6$, there are $5 \times 6 = 30$ pairs of (y,z). Only $(4,5)$, $(4,4)$, $(3,5)$ lead to $x = 9 - y - z \le 1$, and the remaining 27 pairs all lead to a valid solution with $x > 1$. So the number of solutions is 27.

Grading Policy

2 points for replacing x with $x' + 2$ to the equivalent equation; 2 points for $|U|$; 2 points for $|A|$; 2 points for $|B|$; 1 point for $|A \cup B|$; and 1 point for $|U| - |A| - |B| + |A \cap B|$.

Question 11 (10 Points)

Five people check in their hats. They are given back a hat each. Find the number of permutations that no person receives his/her own hat.

Solution

For all the permutations $|U| = 5! = 120$;

Permutations with 1 person receiving his/her hat $|A| = C(5, 1)4! = 120$;

Permutations with 2 persons receiving their hats $|B| = C(5, 2)3! = 60$;

Permutations with 3 persons receiving their hats $|C| = C(5, 3)2! = 20$;

Permutations with 4 persons receiving their hats $|D| = C(5, 4)1! = 5$;

Permutations with 5 persons receiving their hats $|E| = C(5, 5)0! = 1$;

By the inclusion-exclusion principle, the number of permutations that no person receives his/her own hat is

$|U| - |A| + |B| - |C| + |D| - |E| = 120 - 120 + 60 - 20 + 5 - 1 = 44.$

Grading Policy

1 point for correct count on each of the 6 sets, 3 points for correctly applying the inclusion-exclusion principle, and 1 point for the correct result.