CSE 20 Lecture 15

Analysis: Inclusion-Exclusion Technique

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Analysis: Inclusion – Exclusion Principle

Ex: A bakery makes only M=3 kinds of cookies. Find the number of ways a person can buy r=4 of the cookies.

x+y+z = 4 (each variable represents the cookie in that flavor)

C(4+2, 2) = (6*5)/2!= 15

(C(n +m-1, m-1),

n = #cookies,

m = #kinds of cookies)

X	Y	Z
0	0	4
0	1	3
0	2	2
0	3	1
0	4	0
1	0	3
1	1	2
…	…	…

Cont. (from last slide)

Ex: Buy 2 cookies out of 3 flavors

$x+y+z = 2$

$C(2+2, 2) = C(4,2)$

$= (4*3)/2! = 6$

Id	X	Y	Z
0	0	0	2
1	0	1	1
2	0	2	0
3	1	0	1
4	1	1	0
5	2	0	0

Cont. (from last slide)

Ex: Buy 8 cookies out of 4 kinds of flavors.

$x_1+x_2+x_3+x_4 = 8$

C(8+3,3)=11!/(8!3!)=(11*10*9)/3!=165.

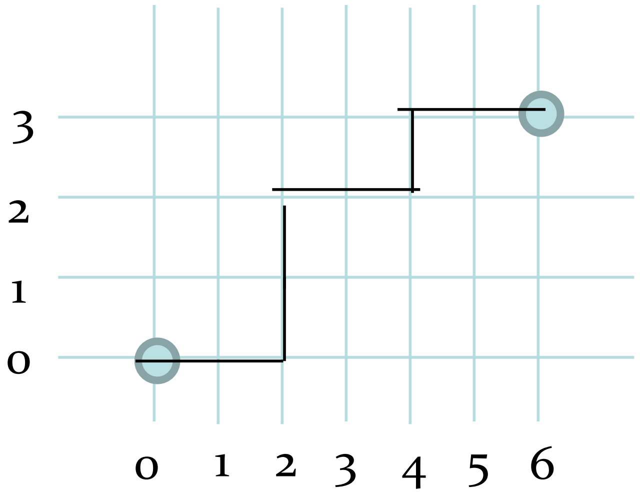
Ex: Buy 20 cookies out of 5 kinds of flavors.

$x_1+x_2+x_3+x_4+x_5 = 20$

C(20+4,4)=24!/(20!4!)=(24*23*22*21)/4!

Ex: Possible number of Routing Paths

Assume that the streets are on a grid and the walking distance must be the shortest. Find the number of ways to walk from (0,0) to (6,3).



View from vertical direction:

$y_0+y_1+y_2+y_3+y_4+y_5+y_6 = 3$

-> $C(3+6, 6) = C(9, 6) = 9!/(6!3!)$

View from horizontal direction:

$x_0+x_1+x_2+x_3 = 6$

-> $C(6+3, 3) = C(9, 6) = 9!/(6!3!)$

Cont. (from last slide)

Ex: On a grid street, find #shortest paths from (1,3) to (2,5).

Formulations: $x_1+x_2 = (5-3)$

Number of combinations: C(2+1,1) = 3!/2!

Ex: On a Manhattan street, find # shortest paths from (-1,-2) to (5,8).

$x_0+x_1+x_2+x_3+x_4+x_5+x_6 = (8-(-2))$

Number of combinations:

C(10+6, 6) = C(16, 6) = 16!/(10!6!)

Integer Linear Equation

Ex: Find the number of integer solutions

$x+y+z = 20$ with $x \geq 4$, $y \geq 5$, $z \geq 6$.

Let $x' = x-4$, $y' = y-5$, $z' = z-6$,

i.e. $x = x'+4$, $y = y'+5$, $z = z'+6$.

Thus, we have an equivalent formulation:

$x'+y'+z' = 5$ with $x', y', z' \geq 0$.

#Combinations: $C(5+2, 2) = C(7,2) =$
$7!/(5!2!) = (7*6)/2 = 21$

Ex: $x+y+z = 20$, $x<7$, $y<8$, $z<9$

U: no constraints. $|U| = C(20+2, 2)$

A: a set of integer solutions with $x \geq 7$

B: a set of integer solutions with $y \geq 8$

C: a set of integer solutions with $z \geq 9$

A: Let $x' = x-7$, i.e. $x = x'+7$ $\quad |A| = C(13+2, 2)$

$x'+y+z = 13$

B: Let $y' = y-8$, i.e. $y = y'+8$ $\quad |B| = C(12+2, 2)$

$x+y'+z = 12$

C: Let $z' = z-9$ $\quad |C| = C(11+2, 2)$

$x+y+z' = 11$

Cont. (from last slide)

$A \cap B$: $x \geq 7$, $y \geq 8$,

Let $x' = x-7$, $y' = y-8$

We have $x'+y'+z = 5$. $|A \cap B| = C(20-7-8+2, 2)$

Likewise, we can derive $|A \cap C| = C(20-7-9+2, 2)$, $|B \cap C| = C(20-8-9+2, 2)$, $|A \cap B \cap C| = 0$.

According to inclusion and exclusion theorem,

the number of solutions is

$|U|-|A|-|B|-|C|+|A \cap B|+|A \cap C|+|B \cap C|-|A \cap B \cap C|$

$= 3$