CSE 20 Lecture Notes

1. Pigeonhole Principle
2. Inclusion-Exclusion Principle

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Pigenhole Principle

A team plays 12 games in a 10 day period and at least one game per day. There is a period of days in which exactly 8 games were played.

Day	0	1	2	3	4	5	6	7	8	9	10
Game	0	1	2	1	1	1	1	1	1	2	1
Si	0	1	3	4	5	6	7	8	9	11	12
Ti	8	9	11	12	13	14	15	16	17	19	20

<u># Pigeons</u>
22

<u># Holes</u>
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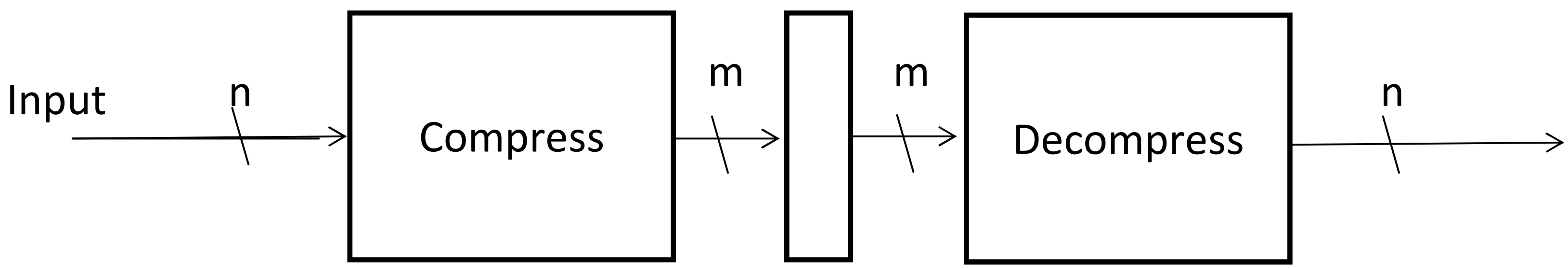
For the case that Si = Tj,
we have Tj = Sj + 8 by definition of Tj.
Thus, Si – Sj = 8.
In other words, 8 games are played from day j+1 to day i

Example 6: Lossless Compression

An algorithm that compress n-bit string to m-bits and then recover the n-bit string



A lossless compression for all possible inputs with n>m does not exist.

# Pigeons	# Holes
2^n	2^m

If n>m, then $2^n > 2^m$

Two different strings map to an identical m-bit storage.

Inclusion-Exclusion Principle

Theorem#1: $|A \bigcup B| = |A| + |B| - |A \bigcap B|.$

Example: Find number of integers in {1 … 100} which is either divisible by 3 or 5.

A: divisible by 3
B: divisible by 5

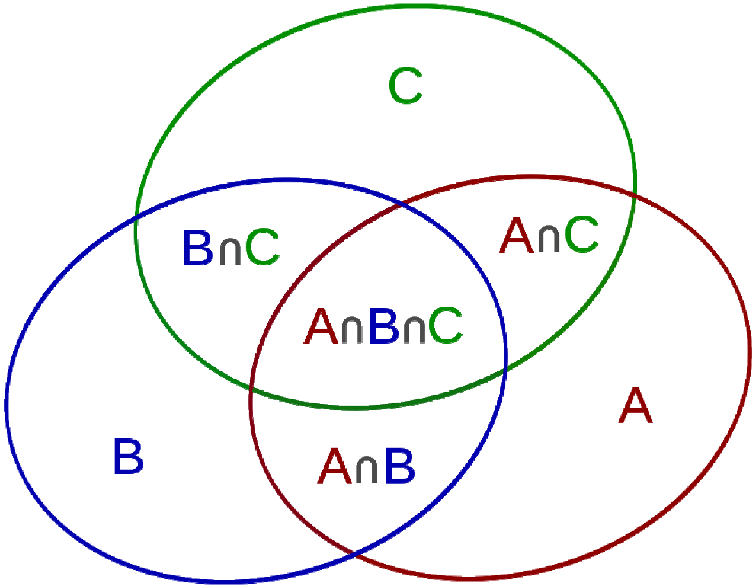
n(A) = [100/3] = 33
n(B) = [100/5] = 20
n(A ∩ B) = [100/15] = 6

$$|A \bigcup B| = |A| + |B| - |A \bigcap B|.$$

= 33 + 20 − 6 = 47

Theorem#2

$$\mathbb{P}(A_1 \cup A_2 \cup A_3) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \mathbb{P}(A_2 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_3)$$



Theorem#3

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i,j\,:\,1\le i<j\le n} |A_i \cap A_j| + \sum_{i,j,k\,:\,1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

Example: Hat Check Problem

n=5 people check in their hats. Later, they are given back their hats. Find the number of ways that no person receives his/her hat.

[Total # of combinations we have.]

5 objects: 1 2 3 4 5

U: # of permutations = 5! (Factorial)

A: Person 1 receives his/her hat, #permutations=4!

“ 2 “ “ “ , …

#rows: 5=C(5,1). |A|= C(5,1)4!.

B: Person 1 & 2 receive their hats, #permutations=3!

Person 1 & 3 receive their hats, …
#rows: C(5,2). |B|= C(5,2)3!.

C: Person 1, 2 & 3 receive their hats, #permutations=2!

Person 1, 2 & 4 receive their hats, …
#rows: C(5,3). |C|= C(5,3)2!.

Example: Hat Check Problem (cont.)

D: Person 1, 2, 3 & 4 receive their hats, #permutations = 1!

Person 1, 2, 3 & 5 receive their hats, …

#rows= C(5,4). |D|=C(5,4)1!.

E: Person 1, 2, 3, 4 & 5 receive their hats, #permutations= 1

#rows= 1. |E|= C(5,5)1.

total # permutations – n(A) + n(B) – n(C) + n(D) – n(E)

$$= 5! - 5! + \frac{5!}{2!} - \frac{5!}{3!} + \frac{5!}{4!} - \frac{5!}{5!}$$

$$= 5! \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \right)$$

$$= 44$$

Include Exclude Include Exclude Include Exclude

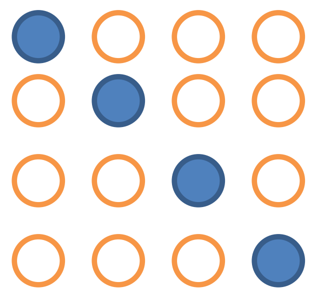
Example 1:

Find the number m of non-negative integer solutions to the equation:

$$x + y = 3$$

4 possible combinations

x	y
0	3
1	2
2	1
3	0



x: # unmarked circles on the left.

y: # unmarked circles on the right.

Example 2:

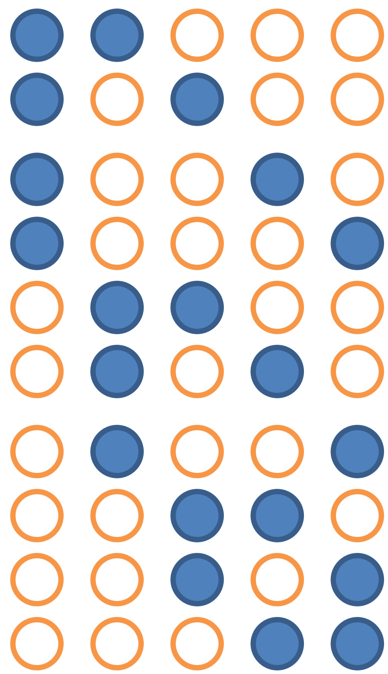
Find the number m of non-negative integer solutions to the equation:

$$x + y + z = 3$$

id	x	y	z
0	0	0	3
1	0	1	2
2	0	2	1
3	0	3	0
4	1	0	2
5	1	1	1
6	1	2	0
7	2	0	1
8	2	1	0
9	3	0	0

NOTE: How to determine # of circles?

value + # of circles being marked.



x: # unmarked circles on the left.

y: # unmarked circles in between.

z: # unmarked circles on the right.

$$c(5, 2) = \frac{5 \times 4}{2} = 10$$

NOTE: # of circles: (w/ 4 variables)

$$x + y + z + w = n$$

$$c(n + 3, 3)$$