CSE 20 Lecture 13 Analysis: Counting with Pigeonhole Principle

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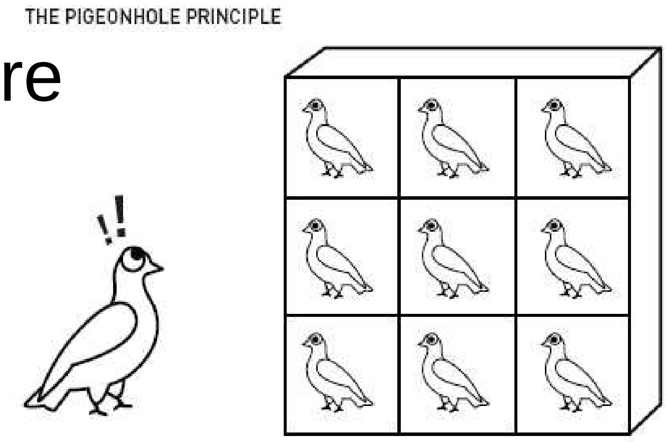
3.3 Pigeonhole Principle



Pigeonhole Principle: If n pigeonholes are occupied by n + 1 or more pigeons, then at least one pigeonhole is occupied by more than one pigeon.

Remark: The principle is obvious. No simpler fact or rule to support or prove it.

Generalized Pigeonhole Principle: If n pigeonholes are occupied by kn + 1 pigeons, then at least one pigeonhole is occupied by k + 1 or more pigeons.

Example 1: Birthmonth

In a group of 13 people, we have 2 or more who are born in the same month.

# pigeons	# holes	At least people born on the same month
13	12	2 or more
20	12	2 or more
121	12	11 or more
65	12	6 or more
111	12	10 or more
kn+1	n	k+1

Example 2: Handshaking

Given a group of n people (n>1), each shakes hands with some (a nonzero number of) people in the group. We can find at least two who shake hands with the same number of people.

Proof:

Number of pigeons (number of people): n

Number of pigeonholes (range of number of handshakes): n-1

Example 3: Cast in theater

A theater performs 7 plays in one season. Five women are each cast in 3 of the plays. Then some play has at least 3 women in its cast.

Number of pigeons (5*3): 15
Number of pigeonholes: 7
k*n+1=2*7+1
3 or more pigeons in the same pigeonhole

Example 4: Pairwise difference

Given 8 different natural numbers, none greater than 14. Show that at least three pairs of them have the same difference.

Try a set: 1, 2, 3, 7, 9, 11, 12, 14
Difference of 12 and 14 = 2.
Same for 9 and 12, 7 and 9, 1 and 3.
In this set, there are four pairs that all have the same difference.
Proof:
Number of pigeons (different pairs: C(8,2) = 8*7/2): 28
Number of pigeonholes (14-1): 13

k*n+1=2*13+1
3 or more pigeons in the same pigeonhole

Example 5: Consecutive game plays

A team plays 12 (g) games in a 10 (n)-day period and at least one game a day. We can always find a period of days in which exactly 8 (m) games are played.

Example set:

days	0	1	2	3	4	5	6	7	8	9	10
games	0	1	1	2	1	1	1	2	1	1	1
s_i	0	1	2	4	5	6	7	9	10	11	12
t_i	8	9	11	12	13	14	15	17	18	19	20

s_i = number of games played from day 1 to day i

$t_i = s_i + 8$

Number of pigeons (number of symbols, $2n+2$): 22

Number of pigeonholes (range, $g+m+1$): 21 (from 0 to 20)

$s_i \neq s_j, \forall\, i \neq j$ $t_i \neq t_j, \forall\, i \neq j$

2 or more pigeons in the same pigeon hole

$s_i = t_j = s_j + 8$, so $s_i - s_j = 8$, 8 games during day $(j+1) \sim i$