CSE 20 Lecture 12

2/25/10

3. Analysis

- 3.1 Introduction
- 3.2 Homogeneous Linear Recursion
- 3.3 Pigeonhole Principle
- 3.4 Inclusion-Exclusion Principle

3.1 Introduction

- Derive the bound of functions or recursions
- Estimate CPU time and memory allocation
- Example on Fibonacci Sequence: We estimate Fn.
  - 0 1 2 3 4 5 6 7 8 9
  - 0 1 1 2 3 4 5 8 13 21 34
  - Fn = 1/r5(((1+rr5)/2)^n-((1-rr5)/2)^n)
    - F0 = 1/rr5(1-1) = 0
    - F1 = 1/rr5(((1+rr5)/2)-((1-rr5)/2)) = 1

$\sqrt{5}$ $\sqrt{5}$ $\sqrt{5}$

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Example: Fibonacci Sequence

0	1	2	3	4	5	6	7	8	9
0	1	1	2	3	5	8	13	21	34

- $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$
- $\approx \frac{1}{2.236}(1.618^n - 0.618^n)$
- $\approx \frac{1}{2.236} \cdot 1.618^n \qquad n \gg 1$
- $n = 9, f_9 \approx 33.98$

3.2 Homogeneous Linear Recursion

- (1) Arithmetic Recursion
  - a, a+d, a+2d, …, a+kd
- (2) Geometric Recursion
  - A, ar, ar^2, …, ar^k
- (3) Linear Recursion
  - $a_n = e_1 a_{n-1} + e_2 a_{n-2} + \ldots + e_k a_{n-k} + f(n)$

Linear Recursion and Homogeneous Linear Recursion

- Linear Recursion: There are no powers or products of $a_j's$
- Homogenous Linear Recursion: A linear recursion with f(n)=0.

Solving Linear Recursion

Input: a_0, a_1, . . ., a_{k-1} where k are the initial values

Express: a_n = e_1 a_{n-1} + . . . + $e_k a_{n-k}$ or a_n - e_1 a_{n-1}- . . . - $e_k a_{n-k}$ = 0

Characteristic Polynomial:

Replace a_n with x^k to get x^k - $e_1 x^{k-1}$ - . . . - e_k = 0

Characteristic Roots: (r_1, . . ., r_k)

Step 1: Find Characteristic Roots (r_1, . . ., r_k)

Set a_{ni} = $\sum_{i=1}^{k} c_i {r_i}^n$ and suppose r_i > r_j such that i > j

Step 2: Determine c_i from k initial values

i.e: a_0, a_1, . . . a_{k-1}

Example 1

Given:

Initial values a_0 = 0 and a_1 = 1

Recursion: a_n = a_{n-1} + a_{n-1}

Rewrite Recursion: a_n- a_{n-1} - a_{n-2} = 0

Characteristic Polynomial: x^2 - *x* – 1 = 0

Characteristic Roots: r_1 = $\frac{1+\sqrt{5}}{2}$ and r_2 = $\frac{1-\sqrt{5}}{2}$

Step 1: a_n = $c_1 r_1$ + $c_2 r_2$

Use initial values n = 0, n =1 for a_n

$$a_0 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0$$

$$a_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1$$

Example 1 (cont'd)

Step 2: Determine c_i

Simplify Equations from Step 1

$a_0 = c_1 1 + c_2 1$

$a_0 = c_1 + c_2$

$0 = c_1 + c_2$ replace a_0 with 0

$a_1 = c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2}$

$1 = c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2}$ replace a_1 with 1

Solve for c_i by replacing c_2 with c_1

$1 = c_1 \frac{1+\sqrt{5}}{2} + c_1 \frac{1-\sqrt{5}}{2}$

$1 = c_1 (\frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2})$

$1 = c_1 (\frac{2\sqrt{5}}{2})$

$1 = c_1 \sqrt{5}$

$c_1 = \frac{1}{\sqrt{5}}$

$0 = c_1 + c_2$

$0 = \frac{1}{\sqrt{5}} + c_2$

$c_2 = -\frac{1}{\sqrt{5}}$

Thus

$$a_n = \frac{\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n - \left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^n}{\sqrt{5}}$$

Example 2

Given:

Initial values a_0 = 1 and a_1 = 1

Recursion: a_n = a_{n-1} + $2a_{n-2}$

Rewrite Recursion: a_n - a_{n-1} - $2a_{n-2}$ = 0

Characteristic Polynomial: x^2 - x – 2 = 0

Characteristic Roots: r_1 = 2 and r_2 = -1

Step 1: a_n = $c_1 r_1$ + $c_2 r_2$

Use initial values n = 0, n =1 for a_n

a_0 = $c_1(2)^0$+ $c_2(-1)^0$

a_1 = $c_1(2)^1$+ $c_2(-1)^1$

Example 2 (cont'd)

Step 2: Determine c_i

Simplify Equations from Step 1

$a_0 = c_1 1 + c_2 1$

$a_0 = c_1 + c_2$

$1 = c_1 + c_2$

$a_1 = c_1 2 + c_2(-1)$

$1 = 2c_1 - c_2$

Solve for c_i with system of equations by replacing c_2 with c_1

$(1 = 2c_1 - c_1) + (1 = c_1 + c_1)$

$2 = 3c_1$

$c_1 = \frac{2}{3}$

$1 = c_1 + c_2$

$1 = \frac{2}{3} + c_2$

$c_2 = \frac{2}{3}$

Thus

$a_n = \frac{2}{3}(2)^n + \frac{1}{3}(-1)^n$

Example 3

For the case that root r is a root of multiplicity w:

$$a_n = c_1 r^n + c_2 n r^n + \ldots + c_w n^w r^n$$

Given:

Initial values $a_0 = 1$ and $a_1 = 1$

Recursion: $a_n = -2a_{n-1} - a_{n-2}$

Rewrite Recursion: $a_n + 2a_{n-1} + a_{n-2} = 0$

Characteristic Polynomial: $x^2 + 2x + 1 = 0$

Characteristic Roots: $r_1 = r_2 = -1$

Step 1: $a_n = c_1 r^n + c_2 n r^n$

$a_0 = c_1(-1^0) + c_2(0)(-1^0)$

$a_1 = c_1(-1^1) + c_2(1)(-1^1)$

Step 2: Match initial values n = 0, n =1, Solve for c_i's

$a_0 = c_1$

$c_1 = 0$

$a_1 = c_1 1 + c_2(-1)$

$1 = 0 + c_2(-1)$ Substitute $c_1 = 0$, $a_1 = 1$

$c_2 = -1$

Thus

$$a_n = c_2 r^n = n(-1)^{n+1}$$

Distinct Roots

For the case that we have u distinct roots, each root r_i

is a root of multiplicity w_i

$$a_n = \sum_{i=1}^{u} g r_i$$

$$g(r_i) = \sum_{j=1}^{w_i} r_{ij} n^j \, (r_i)^n$$