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Student ID number:

prob.	score
1	
2	
3	
4	
5	
6	
7	
total	

1. Show that

$$p \vee ((\sim p \wedge q) \Rightarrow q) \equiv \mathrm{T}.$$

Solution.

$$\begin{aligned} p \vee ((\sim p \wedge q) \Rightarrow q) &\equiv p \vee (\sim(\sim p \wedge q) \vee q) \\ &\equiv p \vee (p \vee \sim q) \vee q \\ &\equiv (p \vee p) \vee (\sim q \vee q) \\ &\equiv p \vee T \\ &\equiv T. \end{aligned}$$

2. We have been studying *inclusive* or: $p \vee q$ means either $p \equiv$ T, $q \equiv$ T, or both. *Exclusive* or, denoted $p \oplus q$, is defined by the following truth table.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

(a) Write a formula for $p \oplus q$ using only $\vee, \wedge, \sim$.
Solution. $(p \wedge \sim q) \vee (\sim p \wedge q)$ works, as does $(p \vee q) \wedge \sim(p \wedge q)$.

(b) Show that $p \oplus p \equiv$ F using the formula from part a.
Solution.

$$\begin{aligned} p \oplus p &\equiv (p \wedge \sim p) \vee (\sim p \wedge p) \\ &\equiv F \vee F \equiv F. \end{aligned}$$

(c) What does $(p \oplus p) \oplus p$ simplify to? You can use part b. Prove your answer.
Solution.

$$\begin{aligned} (p \oplus p) \oplus p &\equiv F \oplus p \\ &\equiv (F \wedge \sim p) \vee (\sim F \wedge p) \\ &\equiv F \vee (\sim F \wedge p) \\ &\equiv F \vee (T \wedge p) \\ &\equiv (T \wedge p) \\ &\equiv p. \end{aligned}$$

3. What is the negation of

$$\forall x \exists y \Big(P(x,y) \Rightarrow \big(Q(x,y) \vee R(x,y) \big) \Big) \quad ?$$

Circle the correct answer and then prove it.

(a) $\forall x \exists y \Big(P(x,y) \wedge {\sim}Q(x,y) \wedge {\sim}R(x,y) \Big)$

(b) $\exists x \forall y \Big(P(x,y) \wedge {\sim}Q(x,y) \wedge {\sim}R(x,y) \Big)$

(c) $\forall x \exists y \Big(P(x,y) \wedge \big(Q(x,y) \vee {\sim}R(x,y) \big) \Big)$

(d) $\exists x \forall y \Big(P(x,y) \wedge \big(Q(x,y) \vee {\sim}R(x,y) \big) \Big)$

Solution.

$$\begin{aligned} {\sim}\Big(\forall x \exists y \Big(P(x,y) \Rightarrow \big(Q(x,y) \vee R(x,y) \big) \Big) \Big) &\equiv \exists x \forall y \;\; {\sim}\Big(P(x,y) \Rightarrow \big(Q(x,y) \vee R(x,y) \big) \Big) \\ &\equiv \exists x \forall y \;\; {\sim}\Big({\sim}P(x,y) \vee \big(Q(x,y) \vee R(x,y) \big) \Big) \\ &\equiv \exists x \forall y \;\; \Big(P(x,y) \wedge {\sim}Q(x,y) \wedge {\sim}R(x,y) \Big). \end{aligned}$$

4. Let n be an integer. Prove that n^2 can be written as either $3k$ or $3k+1$ for some integer k. *Hint*: use the quotient remainder theorem to write n as $3q+r$.
Solution.
By the quotient remainder theorem, $\exists q, r \in \mathbf{Z}$ such that

$$n = 3q + r, \qquad \text{where } 0 \le r < 3.$$

Then $n^2 = (3q+r)^2 = 9q^2 + 6qr + r^2 = 3(3q^2 + 2qr) + r^2$.
<u>Case 1:</u> $r = 0$. Then

$$\begin{aligned} n^2 &= 3(3q^2 + 2qr) + r^2 \\ &= 3(3q^2 + 2qr), \end{aligned}$$

so n^2 can be written as $3k$ with $k = 3q^2 + 2qr$.

<u>Case 2:</u> $r = 1$. Then

$$\begin{aligned} n^2 &= 3(3q^2 + 2qr) + r^2 \\ &= 3(3q^2 + 2qr) + 1, \end{aligned}$$

so n^2 can be written as $3k+1$ with $k = 3q^2 + 2qr$.

<u>Case 3:</u> $r + 2$. Then

$$\begin{aligned} n^2 &= 3(3q^2 + 2qr) + r^2 \\ &= 3(3q^2 + 2qr) + 4 \\ &= 3(3q^2 + 2qr + 1) + 1, \end{aligned}$$

so n^2 can be written as $3k+1$ with $k = 3q^2 + 2qr + 1$.

5. Circle the arguments that are valid.

   (a) Exercise is necessary for my good health.
   I have good health.
   ∴ I exercise.

   (b) If I kick the printer then it will work.
   The printer does not work.
   ∴ I did not kick the printer.

   (c) If they bring the dog then we cannot go to the beach.
   They did not bring the dog.
   ∴ We can go to the beach.

   (d) If Juana is a vegetarian then we will go to Pokez.
   We went to Pokez.
   ∴ Juana is a vegetarian.

   Solution. (a) and (b) are valid.

6. Prove that if $a|b$ and $a|c$ then $a|(bc - 3c)$.
Solution.
Since $a|b$, $\exists x \in \mathbf{Z}$ such that $b = ax$; since $a|c$, $\exists y \in \mathbf{Z}$ such that $c = ay$. Thus

$$bc - 3c = ax(ay) - 3ay = a(axy - 3y),$$

so $a|(bc - 3c)$.

7. (a) Prove that among any 3 consecutive integers, at least one of them is divisible by 3.
Solution.
Consider three successive integers n, $n+1$, and $n+2$. By the quotient-remainder theorem, $n = 3q + r$, where $0 \leq r < 3$. If $r = 0$, $3|n$; if $r = 1$, $n + 2 = 3(q + 1)$, so $3|(n + 2)$; if $r = 2$, $n + 1 = 3(q + 1)$, so $3|(n + 1)$.

(b) Prove that if n is an integer, $n^3 - n$ is divisible by 6. *Hint:* factor $n^3 - n$ into a product of three numbers.
Solution.

$$n^3 - n = n(n^2 - 1) = n(n + 1)(n - 1),$$

so is the product of three successive integers. By part (a), we know that 3 must divide one of those integers and therefore it must divide the product. We learned in class that the product of any two successive integers is even, so the product of 3 successive integers must be even also. In summary: $3|n^3 - n$ and $2|n^3 - n$. Since 2 and 3 have no common factors greater than 1 (they are both prime), we conclude that $6 = 2 \cdot 3$ must divide $n^3 - n$ as well.