CSE 20 - Winter 2008
Midterm Exam: 2008-02-12

Name: ________________________________

Student ID number: ________________________________

prob.	score
1	/6
2	/8
3	/16
4	/12
5	/8
total	/50

Do not start until we announce the start. You will have 80 minutes. No books or calculators are allowed. One double-sided page of notes is allowed.

NOTE: If you need to make an assumption to solve a problem, or use a theorem from class, state the assumption or theorem.

1. (*6 points*)

a) Disprove the following statement:

$$\forall x \in \mathbb{R} \quad \lfloor x^2 \rfloor = \lfloor x \rfloor^2$$

Solution.
Proof: To negate a universal statement, one counter-example will suffice. Let $x = 3/2$. The left-hand-side is $\lfloor x^2 \rfloor = \lfloor 9/4 \rfloor = 2$. The right-hand-side is $\lfloor 3/2 \rfloor^2 = 1^2 = 1$, so the equality does not hold.

b) What is the greatest common divisor (gcd) of 50 and 36?

Solution. $\gcd(50, 36) = 2$.
By factoring, $50 = 2 \cdot 5^2$, and $36 = 6^2 = 2^2 \cdot 3^2$.
So $\gcd(50, 36) = 2^{\min\{1,2\}} \cdot 3^{\min\{0,2\}} \cdot 5^{\min\{2,0\}} = 2^1 \cdot 3^0 \cdot 5^0 = 2$

c) Rewrite the summation,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}$$

using the change of variable, $j = i + 1$.

Solution.

$$\sum_{j=2}^{n+1} \frac{1}{(j-1)j}$$

2. (*8 points*) Suppose that $a_0, a_1, a_2, a_3, \ldots$ is a sequence defined as follows:

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 2 \\ a_2 &= 2 \\ a_j &= a_{j-1} + a_{j-3} \quad \forall j \geq 3 \end{aligned}$$

Prove by induction that a_n is even for all $n \geq 0$.

Solution.
Proof: First we define $P(n)$ as "a_n is even."
Basis step: Since there are three initial conditions, we must prove that the claim holds for each of them.
$P(0)$ holds because $a_0 = 0$, and zero is even because $0 = 2 \cdot 0$.
$P(1)$ holds because $a_1 = 2$, and 2 is even because $2 = 2 \cdot 1$.
$P(2)$ holds because $a_2 = 2$, and 2 is even because $2 = 2 \cdot 1$.

Inductive Step: First we make the inductive hypothesis: assume that for $k \geq 3$, $P(i)$ holds for all $0 \leq i \leq k$. That is, a_i is even.

Now we must prove that $P(k+1)$ is true, i.e. that a_{k+1} is even. By definition of the sequence, $a_{k+1} = a_k + a_{k-2}$. The inductive hypothesis applies to a_k, and also to a_{k-2}, so by definition of even, $\exists m, n \in \mathbb{Z}$ such that $a_k = 2m$ and $a_{k-2} = 2n$. So by subsitution, $a_{k+1} = a_k + a_{k-2} = 2m + 2n = 2(m+n)$. Since $m + n$ is an integer, a_{k+1} is even, by definition of even. $\square$

3. (*16 points*)

a) Prove that $\sqrt{8}$ is irrational. *Hint:* you may use the fact that $\sqrt{2}$ is irrational.

Solution.
Proof by contradiction: Assume not, i.e. assume $\sqrt{8}$ is rational, which means $\sqrt{8} \in \mathbb{Q}$. By the definition of rational, then $\sqrt{8} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}, b \neq 0$. But $\sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$. So then we have that $2\sqrt{2} = \frac{a}{b}$, so $\sqrt{2} = \frac{a}{2b}$. Since $2b \in \mathbb{Z}$ and $2b \neq 0$, because $b \neq 0$, we have that $\sqrt{2} \in \mathbb{Q}$, by definition of rational. We have reached a contradition, because $\sqrt{2}$ is irrational. $\square$

b) Prove that, for $a, b, c \in \mathbb{Z}$, if $a \nmid (b+c)$ then $a \nmid b$ or $a \nmid c$. *Note:* The notation $x \nmid y$ means "x does not divide y."

Solution.
Proof: It will suffice to prove the contrapositive. So we must prove that if $a|b$ and $a|c$, then $a|(b+c)$.
Let $a, b, c \in \mathbb{Z}$, such that $a|b$ and $a|c$. Then by the definition of divisibility, $\exists m, n \in \mathbb{Z}$ such that $b = am$ and $c = an$. So by substitution, $b + c = am + an = a(m+n)$. Since $m + n$ is an integer, $a|(b+c)$, by definition of divisibility. $\square$

Alternate Proof: Proof by contradiction. Assume not. So assume $\exists a, b, c \in \mathbb{Z}$ such that $a \nmid (b+c)$ and $a|b$ and $a|c$. By the definition of divisibility, $\exists m, n \in \mathbb{Z}$ such that $b = am$ and $c = an$. So by substitution, $b + c = am + an = a(m+n)$. Since $m + n$ is an integer, $a|(b+c)$, by definition of divisibility. However we have above that $a \nmid (b+c)$, so we have reached a contridiction. Thus the original claim holds. $\square$

c) Prove that $\forall n \in \mathbb{Z}$, $n^2 - n + 3$ is odd.

Solution.
Proof: Let n be any integer. By the Quotient Remainder theorem, with $d = 2$ (also known as the parity property), n must be either even or odd, so it will suffice to prove the claim in each of those two cases.
Case 1: n is even. By definition of even $n = 2k$ for some $k \in \mathbb{Z}$. So $n^2 - n + 3 = 4k^2 - 2k + 3 = 2(2k^2 - k + 1) + 1$. But $(2k^2 - k + 1)$ is an integer, as it is the sum of products of integers. So $n^2 - n + 3$ is odd, by definition of odd.
Case 2: n is odd. By definition of odd, $n = 2k + 1$ for some $k \in \mathbb{Z}$. So

$$\begin{aligned} n^2 - n + 3 &= (2k+1)^2 - (2k+1) + 3 \\ &= 4k^2 + 4k + 1 - 2k - 1 + 3 \\ &= 4k^2 + 2k + 3 \\ &= 2(2k^2 + k + 1) + 1 \end{aligned}$$

And since $(2k^2 + k + 1) \in \mathbb{Z}$, since it is the sum of products of integers, $n^2 - n + 3$ is odd by definition of odd. $\square$

Note: this can also be proven by contradiction. It can even be proven by induction, but this involves two parts, induction from 0 in the positive direction, and induction from zero in the negative direction.

4. (*12 points*)

a) Negate the following statement:

$$\exists x \forall y \quad \big((P(x,y) \wedge Q(x,y)\big) \rightarrow R(x,y)\big)$$

Circle the correct answer, and then prove it (using equivalences).

1. $\exists x \forall y \quad P(x,y) \wedge Q(x,y) \wedge \sim R(x,y)$
2. $\forall x \exists y \quad P(x,y) \vee Q(x,y) \vee \sim R(x,y)$.
3. $\exists x \forall y \quad \sim P(x,y) \vee \sim Q(x,y) \vee R(x,y)$
4. $\forall x \exists y \quad P(x,y) \wedge Q(x,y) \wedge \sim R(x,y)$.

Solution. 4. $\forall x \exists y \quad P(x,y) \wedge Q(x,y) \wedge \sim R(x,y)$.
Here is a derivation:

$$\begin{aligned}
\sim \Big(\exists x \forall y \quad \big((P(x,y) \wedge Q(x,y)\big) \rightarrow R(x,y)\big)\Big) &\equiv \forall x \quad \sim \Big(\forall y \quad \big((P(x,y) \wedge Q(x,y)\big) \rightarrow R(x,y)\big)\Big) \\
&\equiv \forall x \exists y \quad \sim \Big(\big((P(x,y) \wedge Q(x,y)\big) \rightarrow R(x,y)\big)\Big) \\
&\equiv \forall x \exists y \quad \sim \Big(\sim \big((P(x,y) \wedge Q(x,y)\big) \vee R(x,y)\big)\Big) \\
&\equiv \forall x \exists y \quad \big((P(x,y) \wedge Q(x,y)\big) \wedge \sim R(x,y) \\
&\equiv \forall x \exists y \quad P(x,y) \wedge Q(x,y) \wedge \sim R(x,y)
\end{aligned}$$

b) Circle the arguments that are valid.

1. If Ted visits Las Vegas, then he will gamble.
   Ted did not gamble.
   ∴ Ted did not visit Las Vegas.
2. Being asleep is a necessary condition for dreaming.
   I did not dream.
   ∴ I was not asleep.
3. If it is raining, then we will go to the movies.
   We are going to the movies.
   ∴ It is raining.
4. A sunny day is a sufficient condition for us to go to the beach.
   It is a sunny day.
   ∴ We are going to the beach.

Solution. The valid arguments are 1 and 4. The rest are not valid.

5. (*8 points*)

a) Show the following equivalence, without using a truth table.

$$\sim (p \rightarrow q) \vee (p \wedge q) \equiv p$$

Solution.

$$\begin{aligned} \sim (p \rightarrow q) \vee (p \wedge q) &\equiv \sim (\sim p \vee q) \vee (p \wedge q) \quad \text{equivalence for implies} \\ &\equiv (p \wedge \sim q) \vee (p \wedge q) \quad \text{De Morgan's law} \\ &\equiv p \wedge (q \vee \sim q) \quad \text{distributive law} \\ &\equiv p \wedge T \quad \text{tautology} \\ &\equiv p \quad \text{identity law} \end{aligned}$$

b) The operator $\downarrow$ is defined by the following truth table:

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

Write a logical expression that is equivalent to $P \downarrow Q$, using only $\wedge$, $\vee$, and $\sim$ operators.

Solution. $\sim (P \vee Q)$. Also equivalent is $\sim P \wedge \sim Q$.

c) Draw a digital circuit equivalent to the logical formula you attained in part b), using only AND, OR, and NOT gates.

Solution. Here is one possible diagram, which corresponds to the first answer in b). For the second answer, the two inputs each go through NOT gates, and the outputs go through one AND gate.