Practice Final (Summer 2007) Solutions

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1. Prove that for all sets A and B

$$(A \cup B) \cap (A \cup B^C) = A.$$

Proof: We have to prove containment in both directions.

- $(A \cup B) \cap (A \cup B^C) \subseteq A$:
  Let $x \in (A \cup B) \cap (A \cup B^c)$. By definition of intersection, we have

$$\begin{aligned}(x \in A \cup B) \wedge (x \in A \cup B^c) &\equiv (x \in A \vee x \in B) \wedge (x \in A \vee x \in B^c) \\ &\equiv (x \in A) \vee (x \in B \wedge x \in B^c)\end{aligned}$$

  where the first equivalence is by definition of union, applied to each term, and the second is by the distributive law. Now, assume for the purpose of contradiction that $x \notin A$. Then for the above statement to be true, we must have

$$x \in B \wedge x \in B^c \equiv x \in (B \cap B^c).$$

  But by definition of complement, $B \cap B^c = \emptyset$, so we have a contradiction. Therefore, $x \notin A$ must be false, so $x \in A$.

- $A \subseteq (A \cup B) \cap (A \cup B^C)$:
  Let $x \in A$. Then by definition of union, $x \in A \cup Z$ for any set Z. Let $Z = B$. Then $x \in A \cup Z = A \cup B$. By the same reasoning, we can let $Z = B^c$ and obtain $x \in A \cup B^c$. By definition of intersection, we have

$$x \in (A \cup B) \cap (A \cup B^c).$$

2. The formula for $f(p, q, r)$ can be found by taking the *or* of all rows in the truth table where the output is true:

$$\begin{aligned}
f(p,q,r) &\equiv (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r) & \\
&\equiv r \wedge ((p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)) & \text{by the distributive law on } r \\
&\equiv r \wedge ((q \wedge (p \vee \sim p)) \vee (\sim p \wedge \sim q)) & \text{distributive law on } q \\
&\equiv r \wedge (q \vee (\sim p \wedge \sim q)) & \text{because } (p \vee \sim p) \text{ is a tautology} \\
&\equiv r \wedge ((q \vee \sim p) \wedge (q \vee \sim q)) & \text{distributive law on } q \\
&\equiv r \wedge (q \vee \sim p) & \text{because } (q \vee \sim q) \text{ is a tautology} \\
&\equiv r \wedge (p \rightarrow q) & \text{by definition of } \rightarrow
\end{aligned}$$

3. Show that $\gcd(a, b)$ divides $\text{lcm}(a, b)$, where $a, b \in \mathbb{Z}$. Hint: $\gcd(a, b)$ is the greatest positive integer that divides both a and b; $\text{lcm}(a, b)$ is the least positive integer c such that $a|c$ and $b|c$.

   Proof: Let a, b be any integers, and let $d = \gcd(a, b)$. From the hint, we know that $d|a$. So, for some $m \in \mathbb{Z}$, we have $dm = a$.

   We also know from the hint that for $c = \text{lcm}(a, b)$, $a|c$. So for some $n \in \mathbb{Z}$, $an = c$.

   Putting these two facts together, we have

$$\begin{aligned} an &= \text{lcm}(a, b) = c & \\ (dm)n &= c & \text{by substitution} \\ d(mn) &= c & \\ \gcd(a, b) \cdot mn &= \text{lcm}(a, b) & \text{by substitution} \end{aligned}$$

   Since the integers are closed under multiplication, $mn \in \mathbb{Z}$. Therefore, $\gcd(a, b)|\text{lcm}(a, b)$.

4. Recall that a relation on a set A is a subset of $A \times A$. Suppose that R_1 and R_2 are both equivalence relations on A. Is $R_1 \cap R_2$ necessarily an equivalence relation? Is $R_1 \cup R_2$ necessarily an equivalence relation? Prove your answers.

   Solution: Let A be any set with equivalence relations R_1 and R_2 defined on it.

   - $R_1 \cap R_2$:
     - Reflexive: Since R_1 and R_2 are both reflexive, then $\forall x \in A : (x, x) \in R_1 \wedge (x, x) \in R_2$. Therefore, $(x, x) \in R_1 \cap R_2$, so it is reflexive.
     - Symmetric: Let $(x, y) \in R_1 \cap R_2$. Then $(x, y) \in R_1$ and $(x, y) \in R_2$. Since both R_1 and R_2 are symmetric, we have $(y, x) \in R_1$ and $(y, x) \in R_2$. Therefore, $(y, x) \in R_1 \cap R_2$, and the intersection is symmetric.
     - Transitive: Let $(x, y), (y, z) \in R_1 \cap R_2$. Then since R_1 and R_2 are both transitive, we have $(x, z) \in R_1$ and $(x, z) \in R_2$. So $(x, z) \in R_1 \cap R_2$ and the intersection is transitive.

     This shows that $R_1 \cap R_2$ is an equivalence relation; that is, equivalence relations on a set A are closed under intersection.

   - $R_1 \cup R_2$:
     - Reflexive: Since R_1 is reflexive, then $\forall x \in A : (x, x) \in R_1$. So $(x, x) \in R_1 \cup Z$ for any set Z, so $(x, x) \in R_1 \cup R_2$ and the union is reflexive.
     - Symmetric: Let $(x, y) \in R_1 \cup R_2$. Then there are two cases: either $(x, y) \in R_1$ or $(x, y) \in R_2$. If $(x, y) \in R_1$, then because R_1 is symmetric, $(y, x) \in R_1$. Therefore, $(y, x) \in R_1 \cup R_2$. The same argument can be applied to the case $(x, y) \in R_2$. Therefore, $R_1 \cup R_2$ is symmetric.
     - Transitive: Let $(x, y), (y, z) \in R_1 \cup R_2$. There are two cases to consider here:

* If $(x, y), (y, z)$ come from the same relation (say, they were both in R_1), then we can use the transitivity of the original relation to get (x, z) in the union.
* If (x, y) came from one relation and (y, z) came from the other, eg. $(x, y) \in R_1 - R_2$ and $(y, z) \in R_2 - R_1$, then we cannot use transitivity of the original relations. The following counter-example demonstrates this:
  Let

$$\begin{aligned}
A &= \{1, 2, 3, 4\} \\
R_1 &= \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4)\} \\
R_2 &= \{(1,1), (1,2), (1,4), (2,1), (2,2), (2,4), (3,3), (4,1), (4,2), (4,4)\}
\end{aligned}$$

  Then $R_1 \cup R_2$ has $(3, 1)$ and $(1, 4)$, but not $(3, 4)$. So the union is not transitive.

This shows that the union of equivalence relations is not necessarily an equivalence relation.

5. Prove using induction that

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1 \quad \text{for } n \geq 1.$$

Proof: We define $P(n)$ as the statement: $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$.

- Base case: $n = 1$. [w.t.s. $P(1)$: $\sum_{i=1}^{1} i(i!) = (1+1)! - 1$]

$$\sum_{i=1}^{1} i(i!) = 1 \cdot (1!) = 1$$

  But note that:

$$(1+1)! - 1 = 2 - 1 = 1$$

  So the desired equality holds.

- Induction hypothesis: Fix $n \geq 1$. Assume that the predicate $P(k)$ holds for all k in the range $1 \leq k \leq n$.
- Induction step: We will show $P(n+1)$.

$$\begin{aligned}
\sum_{i=1}^{n+1} i(i!) &= (n+1)(n+1)! + \sum_{i=1}^{n} i(i!) && \text{by splitting out the last term} \\
&= (n+1)(n+1)! + (n+1)! - 1 && \text{by the induction hypothesis} \\
&= (n+1)!((n+1)+1) - 1 && \text{factoring out } (n+1)! \\
&= (n+2)(n+1)! - 1 = (n+2)! - 1 && \text{by definition of factorial} \\
&= ((n+1)+1)! - 1.
\end{aligned}$$

6. Suppose that $A = \{A_1, A_2, \ldots, A_m\}$ and $B = \{B_1, B_2, \ldots, B_n\}$ are each partitions of a set X. Prove that

$$P = \{A_i \cap B_j | i = 1, \ldots, m,\ j = 1, \ldots, n\}$$

is also a partition of X.

Proof: Let A, B, and P be defined as above. To verify that P is a partition, we must prove two things. For any $x \in X$, we must show that it belongs to some subset of P. We must also show that each $x \in X$ belongs to no more than one subset of P.

Let $x \in X$. Then since A and B are both partitions of X, there exists some subsets $A_i \in A$ and $B_j \in B$ such that $x \in A_i$ and $x \in B_j$. Then by definition of intersection, $x \in A_i \cap B_j$. Since P is the collection of all such intersections, $A_i \cap B_j \in P$. Therefore, every $x \in X$ belongs to at least one subset of P.

Now, assume toward a contradiction that there are two distinct subsets of P containing some $x \in X$. That is,

$$\exists x \in X,\ \exists i, j, k, \ell :\ (x \in A_i \cap B_j) \wedge (x \in A_k \cap B_\ell) \wedge (i \neq k \vee j \neq \ell)$$

Then by definition of intersection, we have $x \in A_i \cap B_j \cap A_k \cap B_\ell$. Rearranging this, we get $x \in (A_i \cap A_k) \cap (B_j \cap B_\ell)$. If $i \neq k$, then because A is a partition, $A_i \cap A_k = \emptyset$. Similarly, if $j \neq \ell$, $(B_j \cap B_\ell) = \emptyset$. In either event, we have $x \in \emptyset$, which is a contradiction. Therefore, there cannot be two distinct subsets of P which contain x.

7. Let $f : A \to B$ and $g : B \to C$. Prove that if $g \circ f$ is onto then g must be onto. Give an example to show that f does *not* have to be onto when $g \circ f$ is.

- **Proof:** Let $f : A \to B$ and $g : B \to C$, such that $g \circ f$ is onto. Assume toward a contradiction that g is not onto. That is,

  $$\exists y \in C,\ \forall x \in B,\ g(x) \neq y.$$

  But, since $g \circ f$ is onto,

  $$\forall y \in C,\ \exists z \in A,\ g(f(z)) = y.$$

  Let $x = f(z)$. By definition f, $x \in B$. This gives us our contradiction, because for this x, $g(x) = y$.

- Let $A = \{1, 2\}, B = \{1, 2\}, C = \{1\}$. Define $f : A \to B$ and $g : B \to C$ as follows: $f(x) = 1$ and $g(x) = 1$. In other words, both functions are the constant value 1, but each one has a different co-domain. The function $g \circ f$ is onto, because it is defined for the only element in its co-domain, C, (the element is 1). But f is not onto because $2 \in B$, but there is no $x \in A$ such that $f(x) = 2$.