

Midterm I
CSE 21, Winter 2006

Name:

Student ID:

Conditions: No books, no consultation, nor any kind of assistance allowed. You can bring one page of notes (both sides). Be sure to show your work where indicated.

1. (20 pts)

- (a) How many ways can the letters of the word THEORY be arranged in a row?
- (b) How many ways can the letters of the word THEORY be arranged in a row if T and H must remain next to each other as either TH or HT?

Answer:

(a) #permutations of six letters = $6! = 720$.

(b)

Step 1: Treat T and H as a group. Arrange the T&H group and the other four letters in a row. #permutations of five elements = $5!$.

Step 2: Place T and H. There are two ways to place T and H: TH and HT.

Finally: by multiplication rule, we have the solution = $2 \cdot 5! = 240$.

2. (20 pts)

Suppose that we only have two kinds of coins, a 4 ¢ coin and a 5 ¢ coin.

Prove by induction that for all integers $n \geq 12$, n cents can be formed using 4 ¢ and 5 ¢ coins.

Answer:

Statement $P(n)$: n cents can be formed using 4 ¢ and 5 ¢ coins, for an integer $n \geq 12$.

(i) Basis.

12 cents can be formed using three 4 ¢ and zero 5 ¢ coins, so $P(n = 12)$ is true;

13 cents can be formed using two 4 ¢ and one 5 ¢ coins, so $P(n = 13)$ is true;

14 cents can be formed using one 4 ¢ and two 5 ¢ coins, so $P(n = 14)$ is true; and

15 cents can be formed using zero 4 ¢ and three 5 ¢ coins, so $P(n = 15)$ is true.

(ii) Inductive Step.

Assume that $P(n)$ is true for all integers n when $12 \leq n \leq k$ and $k \geq 15$.

When $n = k + 1$, we know that $n = 4 + (k - 3)$ and $P(k - 3)$ is true according to the assumption, since $12 \leq k - 3 \leq k$. Therefore, $P(n = k + 1)$ is true.

(iii) Conclusion.

We conclude that $P(n)$ is true for all integers $n \geq 12$.

Alternative answer:

(i) Basis.

12 cents can be formed using three 4 ¢ and zero 5 ¢ coins, so $P(n = 12)$ is true.

(ii) Inductive Step.

Assume that $P(n)$ is true for $12 \leq k$. We want to show that $P(n+1)$ is true. We consider two cases.

Case (i): The solution of $P(n)$ has at least one 4 ¢ coin. To get $n+1$ we replace one 4 ¢ coin by one 5 ¢ coin, which will increase the total value by 1 ¢.

Case (ii): The solution of $P(n)$ has only 5 ¢ coins. Since $n \geq 12$ and is divisible by 5 there are at least three 5 ¢ coins. To get $n+1$ we replace three 5 ¢ by four 4 ¢, which will increase the total value by 1 ¢.

(iii) Conclusion.

We conclude that $P(n)$ is true for all integers $n \geq 12$.

3. (20 pts)

Five cards are selected at random from an ordinary deck of 52 cards. What is the probability that all cards are from the same suit (i.e., a *flush*)?

Answer:

(1) Total #combinations of selecting 5 cards from a deck of 52 cards = $\binom{52}{5}$.

(2) Total #combinations of selecting 5 cards from one suit of 13 cards = $\binom{13}{5}$.

Since there are 4 suits, the total #combinations of selecting 5 cards from the same suit = $4 \cdot \binom{13}{5}$.

(3) Probability of selecting 5 cards from the same suit = $\frac{4 \cdot \binom{13}{5}}{\binom{52}{5}}$

4. (20 pts)

In a group of 100 students, 60 like pizza, 56 like spaghetti, 40 like lasagna. Furthermore, 35 like both pizza and spaghetti, 25 like both spaghetti and lasagna, 78 don't like both pizza and lasagna, 10 like all three. How many students like none of these three foods?

Answer:

Let ALL be the set of 100 students,

P be the set of students who like pizza,

S be the set of students who like spaghetti, and

L be the set of students who like lasagna.

From the question, we have

$$N(ALL) = 100,$$

$$N(P) = 60,$$

$$N(S) = 56,$$

$$N(L) = 40,$$

$$N(P \cap S) = 35,$$

$$N(S \cap L) = 25,$$

$$N(\overline{P \cap S}) = 78, \text{ and}$$

$$N(P \cap S \cap L) = 10.$$

Therefore, we know that

$$N(P \cap S) = N(ALL) - N(\overline{P \cap S}) = 100 - 78 = 22,$$

$$N(P \cup S \cup L) = N(P) + N(S) + N(L) - N(P \cap S) - N(S \cap L) - N(P \cap L) + N(P \cap S \cap L) = 60 + 56 + 40 - 35 - 25 - 22 + 10 = 84,$$

and finally the number of students who like none of these three foods $N(\overline{P \cup S \cup L}) = N(ALL) - N(P \cup S \cup L) = 100 - 84 = 16$.

5. (20 pts)

Suppose we have 10 blue balls, 15 red balls and 18 green balls. How many ways can we place all these balls into 3 bins?

Answer:

Notice that the assignment of balls of one color is independent of the assignment of balls of the other two colors. Therefore, the number of ways of place 10 blue balls, 15 red balls and 18 green balls into 3 bins = $\binom{10+2}{2} \cdot \binom{15+2}{2} \cdot \binom{18+2}{2}$

$$= \binom{12}{2} \cdot \binom{17}{2} \cdot \binom{20}{2}.$$