

# Knapsack Problem

A knapsack with weight-carrying capacity  $W$ .

$\exists$  unlimited supply of  $n$  types of items with weight  $w_i > 0$ , and value

$$v_i \geq 0.$$

Goal: Get the maximum value without breaking the knapsack.

# Knapsack Problem

Maximize  $V = \sum_j v_j x_j$

Subject to  $\sum_j w_j x_j \leq W$

$x_j \geq 0$ , integers

$x_j$  is the number of items selected of the  $j$ th type.

- Model?

- $x_j$  not integers?

## If $x_j$ are not Integers

Order the items such that:

$$\frac{v_1}{w_1} > \frac{v_2}{w_2} > \dots > \frac{v_n}{w_n}$$

Then the solution is  $x_1 = \left( \frac{W}{w_1} \right)$ , with  
value  $x_1 v_1$ .

$$\text{Max } 12x_1 + 10x_2 + 1x_3$$

$$\text{Sub } 11x_1 + 10x_2 + 9x_3 \leq 20$$

$$x_j \geq 0$$

# Max Return Function

Define Max Return Function  $F_k(y)$ :

Max value if the capacity is  $y$   
( $y = 1, 2, \dots, W$ ), where only the  
first  $k$  types of items are available.

$$F_k(0) = 0$$

$$F_1(y) = \left\lfloor \frac{y}{w_1} \right\rfloor v_1$$

# Max Return Function

$$F_k(y) = \max \begin{cases} F_{k-1}(y) \\ v_k + F_k(y - w_k) \end{cases}$$

| $k \backslash y$ | 0 | 1 | 2 | 3 | 4 | ... | $b$ |
|------------------|---|---|---|---|---|-----|-----|
| 1                |   |   |   |   |   |     |     |
| 2                |   |   |   |   |   |     |     |
| 3                |   |   |   |   |   |     |     |
| $\vdots$         |   |   |   |   |   |     |     |

# Complexity

If there are  $n$  types of items, and  $W = b$ , then the total space is  $O(nb)$ , and the total time is  $O(nb)$ .

Is this a linear algorithm?  
Reasonable encoding  $b = 32$ .

# Example

| $b$ | $x_1$ | $x_2$ | $x_3$ | $W$ | $V$ | $G$ |
|-----|-------|-------|-------|-----|-----|-----|
| 1   | 0     | 0     | 0     | 0   | 0   | 0   |
| 2   | 0     | 0     | 1     | 2   | 1   | 1   |
| 3   | 0     | 0     | 1     | 2   | 1   | 1   |
| 4   | 0     | 1     | 0     | 4   | 3   | 3   |
| 5   | 1     | 0     | 0     | 5   | 5   | 0   |
| 6   | 1     | 0     | 0     | 5   | 5   | 0   |
| 7   | 1     | 0     | 1     | 7   | 6   | 1   |
| 8   | 1     | 0     | 1     | 7   | 6   | 1   |
| 9   | 1     | 1     | 0     | 9   | 8   | 3   |
| 10  | 2     | 0     | 0     | 10  | 10  | 0   |
| 11  | 2     | 0     | 0     | 10  | 10  | 0   |
| 12  | 2     | 0     | 1     | 12  | 11  | 1   |
| 13  | 2     | 0     | 1     | 12  | 11  | 1   |
| 14  | 2     | 1     | 0     | 14  | 13  | 3   |
| 15  | 3     | 0     | 0     | 15  | 15  | 0   |