

1 Jan 17, 2006

1.1 Floyd Warshall Algorithm

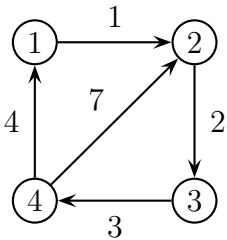
- Finds shortest paths between all pairs of nodes
- $d_{i,j} \geq 0$, but no negative cycles
- $d_{i,k} \geq d_{i,j} + d_{j,k}$

$$d_{i,k} \leftarrow \min(d_{i,k}, d_{i,j} + d_{j,k})$$

For $j = 1, 2, \dots, n$ and all $i, k \neq j$

After $j = n$, we have the shortest distance between any pair of nodes.

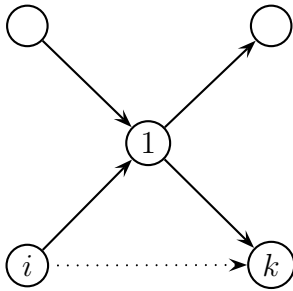
1.2 Numerical Example



	①	②	③	④
①	0	1	∞	∞
②	∞	0	2	∞
③	∞	∞	0	3
④	4	7	∞	0

1.3 Another Example

When $j = 1 \dots$

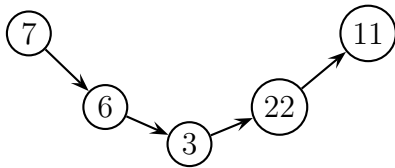


If $d_{i,k} \leq d_{i,1} + d_{1,k}$ then $d_{i,k}$ remains unchanged

If $d_{i,k} > d_{i,1} + d_{1,k}$ then $d_{i,k}$ is updated with the sum

Then try $j = 2$, update with smaller values

1.4 Basic Arcs



If $(7) \sim (11)$ is shortest

then $(6) \sim (22)$ is shortest

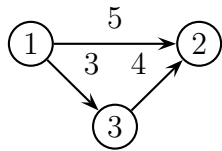
then $(6) \sim (3)$ is shortest

Basic Arc (i, j) : An arc (i, j) is a basic arc iff the shortest path from (i) to (j) is the arc (i, j) .

S1: A shortest path must consist of basic arcs. (True or False)

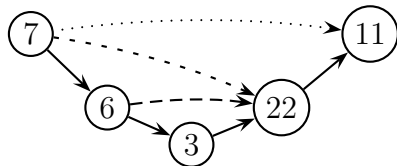
S2: A path consisting of basic arcs must be shortest. (True or False)

Counterexample to S2:

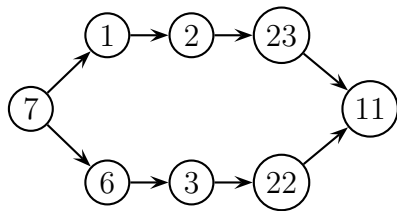


Proof of S1: Assuming no ties

If we focus on an arbitrary shortest path, and we can get the shortest distance, then it is correct for all



1.5 Ties...



In case of tie, which path will be picked by the computer?

1.6 We want the path also

We got the shortest distance, but not the shortest path

$p_{i,k} = k$ for all i, k to start

We have an associated array:

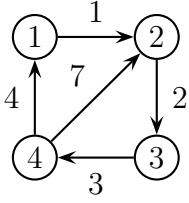
	①	②	③	④
①	1	2	3	4
②	1	2	3	4
③	1	2	3	4
④	1	2	3	4

$$p_{i,k} = \begin{cases} j & \text{if } d_{i,k} > d_{i,j} + d_{j,k} \\ \text{same} & \text{if } d_{i,k} \leq d_{i,j} + d_{j,k} \end{cases}$$

This gives us *some* node j between i and k , but we want the *first* node between i and k .

Change the algorithm to:

$$p_{i,k} = \begin{cases} p_{i,j} & \text{if } d_{i,k} > d_{i,j} + d_{j,k} \\ \text{same} & \text{if } d_{i,k} \leq d_{i,j} + d_{j,k} \end{cases}$$



$$\begin{aligned} j = 1 : & \quad p_{4,2} \leftarrow p_{4,1} \leftarrow 1 \\ j = 2 : & \quad p_{1,3} \leftarrow p_{4,2} \leftarrow p_{4,1} \leftarrow 1 \\ j = 3 : & \quad p_{4,3} \leftarrow p_{4,2} \leftarrow 1 \end{aligned}$$

$$\begin{aligned} \textcircled{4} \sim \textcircled{3} &= p_{4,3} = \textcircled{1} \\ \textcircled{1} \sim \textcircled{3} &= p_{1,3} = \textcircled{2} \\ \textcircled{2} \sim \textcircled{3} &= p_{2,3} = \textcircled{3} \end{aligned}$$

$d_{i,k}$:

	$\textcircled{1}$	$\textcircled{2}$	$\textcircled{3}$	$\textcircled{4}$
$\textcircled{1}$	0	1	∞	∞
$\textcircled{2}$	∞	0	2	∞
$\textcircled{3}$	∞	∞	0	3
$\textcircled{4}$	4	7	∞	0

$j = 1$:

	$\textcircled{1}$	$\textcircled{2}$	$\textcircled{3}$	$\textcircled{4}$
$\textcircled{1}$				
$\textcircled{2}$				
$\textcircled{3}$				
$\textcircled{4}$		5		

$$p_{4,2} \leftarrow p_{4,1} = 1$$

$j = 2$:

	①	②	③	④
①	0		3	
②		0		
③			0	
④		5	7	0

$p_{1,3} \leftarrow p_{1,2} = 2$

1.7 Implementation

Given n by n array

$j = 1$:

	①	②	③	④
①				
②				
③				
④				

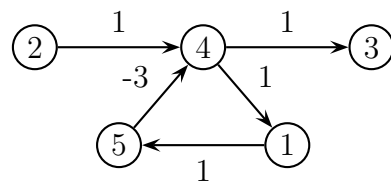
Update $(n - 1)$ by $(n - 1)$ unshaded region.

$j = 1$:

	①	②	③	④
①				
②				
③				
④				

Update unshaded region.

1.8 Negative cycles?



How to find negative cycle?

Set $d_{i,i} = \infty$ to start, then run Floyd Warshall

