CSE 200 Homework 5 Solutions

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Claim 1. *Let $F \in$ 2-SAT and let G be the digraph where $V(G)$ is the literals of F and there is a directed edge from a to b iff $\{\overline{a}, b\} \in F$. Then F is satisfiable iff G has no cycle involving a literal and its negation.*

Proof. ($\Rightarrow$) $\overline{a} \vee b$ is equivalent to $a \to b$. Suppose there is a path from a to $\overline{a}$ to a in G. Then by transitivity, the satisfiability of F would imply that $a \leftrightarrow \overline{a}$, a contradiction.

($\Leftarrow$) We use induction on the number of variables. If F has no variables, then F is satisfiable, and we are done. Otherwise let a be some literal in G from which $\overline{a}$ cannot be reached.

Suppose indirectly that there is a path from a to both some variable b and its negation $\overline{b}$. Then by contraposition, there is also a path from $\overline{b}$ to $\overline{a}$, and by transitivity, there is a path from a to $\overline{a}$, a contradiction.

Set all of the literals reachable from a in G to true. This is well-defined since no literal and its negation can both be reached from a. Furthermore, all clauses involving the literals just set are satisfied, and the graph G' on the remaining formula also fails to have a cycle involving a literal and its negation. By the inductive hypothesis, the remaining formula is satisfiable, and we are done. □

Claim 2. 2-SAT $\in$ NL.

Proof. Since NL = coNL, it is sufficient to show that $\overline{\text{2-SAT}} \in$ NL. We will demonstrate a log-space reduction from $\overline{\text{2-SAT}}$ to PATH $\in$ NL. So let $F \in$ 2-CNF have variable set X and literal set L. Let G be the digraph defined in claim 1.

For each $l \in L$, create a new copy G_l of G:

$$V(G_l) = V(G) \times \{l\}$$
$$E(G_l) = \{((a,l),(b,l)) \mid (a,b) \in E(G)\}$$

and let s, t be 2 nodes not in $V' = \cup_{l \in L} V(G_l)$. Define digraph H by $V(H) = V' \cup \{s,t\}$ and

$$\begin{aligned} E(H) = & \cup_{l \in L} E(G_l) \\ & \cup \{(s,(x,x)) \mid x \in X\} \\ & \cup \{((\overline{x},x),(\overline{x},\overline{x})) \mid x \in X\} \\ & \cup \{((x,\overline{x}),t) \mid x \in X\}. \end{aligned}$$

The idea is that we can arrange the G_l into 2 rows: the 1st row for those G_x where $x \in X$ and the 2nd row for those $G_{\overline{x}}$ where $x \in X$. s is above the 1st row and t is below the 2nd row. For each $x \in X$, there is an edge from s to the copy of x in G_x, an edge from the copy of $\overline{x}$ in G_x to the copy of $\overline{x}$ in $G_{\overline{x}}$, and an edge from the copy of x in $G_{\overline{x}}$ to t.

We now show that there is an $x \in X$ and a path from x to $\overline{x}$ to x in G iff there is a path from s to t in H.

($\Rightarrow$) Let p be a path in G from x to $\overline{x}$ to x. Then the corresponding path in H goes from s to the copy of x in G_x, follows the isomorphic copy of p until reaching the copy of $\overline{x}$, then takes the edge from the copy of $\overline{x}$ in G_x to the copy of $\overline{x}$ in $G_{\overline{x}}$, then continues to follow the isomorphic copy of p until reaching the copy of x, and finally takes the edge from the copy of x in $G_{\overline{x}}$ to t.

($\Leftarrow$) If p is a path from s to t in H, then it must first enter G_x for some $x \in X$. By stripping off s, t from p and the edge from G_x to $G_{\overline{x}}$ and removing the 2nd component information in the nodes of p, we obtain a path in G from x to $\overline{x}$ to x.

So, by claim 1, $F \mapsto H$ is a reduction from $\overline{\text{2-SAT}}$ to PATH. The following algorithm shows that H can be computed in log space:

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1  for each l ∈ L and {l1, l2} ∈ F
2    output ((¬l1, l), (l2, l)), ((¬l2, l), (l1, l))
3  for each x ∈ X
4    output (s, (x, x)), ((x̄, x), (x̄, x̄)), ((x, x̄), t)
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□

Claim 3. 2-SAT $\in$ NL-hard.

Proof. We show a log-space reduction from $\overline{\text{PATH}}$ to 2-SAT which essentially encodes the phrase "there is a cut separating s from t". Let $G = (V, E)$ be a digraph and $s, t \in V$. For each $(x, y) \in E$, add the clause $\{\overline{x}, y\}$ to F. Also add the unit clauses $\{s\}, \{\overline{t}\}$ to F. Clearly F can be computed in log space. It remains to show that $(G, s, t) \in \overline{\text{PATH}}$ iff $F \in$ 2-SAT.

($\Rightarrow$) If there is no path from s to t, then there is a cut separating s from t. Set the variables in the cut to true and the others to false to satisfy F.

($\Leftarrow$) If F is satisfied by some assignment a, then the true variables in a form a cut separating s from t and so there is no path from s to t. □