

CSE 200 Homework 5 Solutions

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Claim 1. *Let $F \in 2\text{-SAT}$ and let G be the digraph where $V(G)$ is the literals of F and there is a directed edge from a to b iff $\{\bar{a}, b\} \in F$. Then F is satisfiable iff G has no cycle involving a literal and its negation.*

Proof. $(\Rightarrow)$ $\bar{a} \vee b$ is equivalent to $a \rightarrow b$. Suppose there is a path from a to $\bar{a}$ to a in G . Then by transitivity, the satisfiability of F would imply that $a \leftrightarrow \bar{a}$, a contradiction.

$(\Leftarrow)$ We use induction on the number of variables. If F has no variables, then F is satisfiable, and we are done. Otherwise let a be some literal in G from which $\bar{a}$ cannot be reached.

Suppose indirectly that there is a path from a to both some variable b and its negation $\bar{b}$. Then by contraposition, there is also a path from $\bar{b}$ to $\bar{a}$, and by transitivity, there is a path from a to $\bar{a}$, a contradiction.

Set all of the literals reachable from a in G to true. This is well-defined since no literal and its negation can both be reached from a . Furthermore, all clauses involving the literals just set are satisfied, and the graph G' on the remaining formula also fails to have a cycle involving a literal and its negation. By the inductive hypothesis, the remaining formula is satisfiable, and we are done. $\square$

Claim 2. $2\text{-SAT} \in \text{NL}$.

Proof. Since $\text{NL} = \text{coNL}$, it is sufficient to show that $\overline{2\text{-SAT}} \in \text{NL}$. We will demonstrate a log-space reduction from $\overline{2\text{-SAT}}$ to $\text{PATH} \in \text{NL}$. So let $F \in 2\text{-CNF}$ have variable set X and literal set L . Let G be the digraph defined in claim 1.

For each $l \in L$, create a new copy G_l of G :

$$\begin{aligned} V(G_l) &= V(G) \times \{l\} \\ E(G_l) &= \{((a, l), (b, l)) \mid (a, b) \in E(G)\} \end{aligned}$$

and let s, t be 2 nodes not in $V' = \cup_{l \in L} V(G_l)$. Define digraph H by $V(H) = V' \cup \{s, t\}$ and

$$\begin{aligned} E(H) &= \cup_{l \in L} E(G_l) \\ &\cup \{(s, (x, x)) \mid x \in X\} \\ &\cup \{((\bar{x}, x), (\bar{x}, \bar{x})) \mid x \in X\} \\ &\cup \{((x, \bar{x}), t) \mid x \in X\}. \end{aligned}$$

The idea is that we can arrange the G_l into 2 rows: the 1st row for those G_x where $x \in X$ and the 2nd row for those $G_{\bar{x}}$ where $x \in X$. s is above the 1st row and t is below the 2nd row. For each $x \in X$, there is an edge from s to the copy of x in G_x , an edge from the copy of $\bar{x}$ in G_x to the copy of $\bar{x}$ in $G_{\bar{x}}$, and an edge from the copy of x in $G_{\bar{x}}$ to t .

We now show that there is an $x \in X$ and a path from x to $\bar{x}$ to x in G iff there is a path from s to t in H .

($\Rightarrow$) Let p be a path in G from x to $\bar{x}$ to x . Then the corresponding path in H goes from s to the copy of x in G_x , follows the isomorphic copy of p until reaching the copy of $\bar{x}$, then takes the edge from the copy of $\bar{x}$ in G_x to the copy of $\bar{x}$ in $G_{\bar{x}}$, then continues to follow the isomorphic copy of p until reaching the copy of x , and finally takes the edge from the copy of x in $G_{\bar{x}}$ to t .

($\Leftarrow$) If p is a path from s to t in H , then it must first enter G_x for some $x \in X$. By stripping off s, t from p and the edge from G_x to $G_{\bar{x}}$ and removing the 2nd component information in the nodes of p , we obtain a path in G from x to $\bar{x}$ to x .

So, by claim 1, $F \mapsto H$ is a reduction from $\overline{2\text{-SAT}}$ to PATH. The following algorithm shows that H can be computed in log space:

- 1 for each $l \in L$ and $\{l_1, l_2\} \in F$
- 2 output $((\bar{l}_1, l), (l_2, l)), ((\bar{l}_2, l), (l_1, l))$
- 3 for each $x \in X$
- 4 output $(s, (x, x)), ((\bar{x}, x), (\bar{x}, \bar{x})), ((x, \bar{x}), t)$

□

Claim 3. $2\text{-SAT} \in \text{NL-hard}$.

Proof. We show a log-space reduction from $\overline{\text{PATH}}$ to 2-SAT which essentially encodes the phrase “there is a cut separating s from t ”. Let $G = (V, E)$ be a digraph and $s, t \in V$. For each $(x, y) \in E$, add the clause $\{\bar{x}, y\}$ to F . Also add the unit clauses $\{s\}, \{\bar{t}\}$ to F . Clearly F can be computed in log space. It remains to show that $(G, s, t) \in \overline{\text{PATH}}$ iff $F \in 2\text{-SAT}$.

($\Rightarrow$) If there is no path from s to t , then there is a cut separating s from t . Set the variables in the cut to true and the others to false to satisfy F .

($\Leftarrow$) If F is satisfied by some assignment a , then the true variables in a form a cut separating s from t and so there is no path from s to t . □