CSE 200 Homework 4 Solutions

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1 NAE-SAT

Define NAE-SAT as those Boolean formulas in conjunctive normal form for which there is an assignment that makes each clause have at least 1 true literal and at least 1 false literal (i.e. in each clause, not all literals are equal).

Claim 1. NAE-SAT $\in$ NP-*complete.*

Proof. Clearly NAE-SAT $\in$ NP. To show hardness, we reduce from SAT: let $F \in \text{CNF}$ be a set of clauses, where a clause is a set of literals, and let z be a variable not appearing in F. Let $F' = \{C \cup \{z\} \mid C \in F\}$. Clearly F' can be constructed in polynomial time. It remains to show that $F \in \text{SAT}$ iff $F' \in \text{NAE-SAT}$.

($\Rightarrow$) If a is a satisfying assignment to F, then we can extend a by setting z to 0 to get a NAE satisfying assignment to F'.

($\Leftarrow$) If a is a NAE satisfying assignment to F', then, since the complement of a is also a NAE satisfying assignment to F', we may assume wlog that z is set to 0 in a. The restriction of a to the variables of F is a satisfying assignment to F. $\square$

This proves that NAE-4-SAT is NP-complete. It is interesting to note that we can also prove that NAE-3-SAT is NP-complete by reducing from 3-COLORING to 3-SAT, and then further reduce to NAE-3-SAT, by adding z to the clauses of width 2.

2 SET-SPLITTING

Define

$$\text{SET-SPLITTING} = \{C \mid \text{ each } c \in C \text{ is finite and } \exists A \subseteq \bigcup C \; \forall c \in C \; (A \cap c \neq \emptyset \wedge \overline{A} \cap c \neq \emptyset)\}.$$

Claim 2. SET-SPLITTING $\in$ NP-*complete.*

Proof. Clearly SET-SPLITTING $\in$ NP since a certificate is such an A. To prove hardness we reduce from NAE-SAT: let $F \in \text{CNF}$. Set $C = F \cup \{\{x, \overline{x}\} \mid x \text{ is a variable of } F\}$. Clearly C can be constructed in polynomial time. It remains to show that $F \in \text{NAE-SAT}$ iff $C \in \text{SET-SPLITTING}$.

($\Rightarrow$) If a is a NAE satisfying assignment to F, then set A to be the true literals. Since each clause $c \in F$ contains a true literal and a false literal and since for each variable x exactly one of $x, \overline{x}$ is true, we have that A satisfies C.

($\Leftarrow$) If A is a solution to C, then for each variable x, set x to true iff $x \in A$. Since exactly one of $x, \overline{x}$ is in A, we have that x is false iff $\overline{x} \in A$. So each literal is true iff it is in A. Since each clause $c \in F$ contains an element of A and an element of $\overline{A}$, c must contain a true literal and a false literal. So $F \in \text{NAE-SAT}$. $\square$