CSE 200 Homework 3 Solutions

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Let L be the following problem: given digraph $G = (V, E)$ and $S \subseteq V^2$, decide whether there is a path p from s to t s.t. $\forall (u, v) \in S$, p uses at most one of u, v.

Claim 1. $L \in$ NP-complete.

Proof. $L \in$ NP since a witness is such a path p and if there is any such path then there is one of length at most $|V|$. A path of polynomial size can easily be verified in polynomial time.

To show that L is NP-hard, we reduce from 3-SAT: Let $F = \{C_1, \ldots, C_m\} \in$ 3-CNF be a set of m 3-clauses, where a 3-clause is a set of ≤ 3 literals. Let $V' = \{(l, C) \mid l \in C \in F\}$ and $V = V' \cup \{s, t\}$ where $s, t \notin V', s \neq t$. So V is the set of literal occurrences of F, together with 2 new nodes s, t. Let

$$\begin{aligned} E = \{((k, C_i), (l, C_{i+1})) \in V^2 \mid i \in \{1, \ldots, m-1\}\} \\ \cup \{(s, (l, C_1)) \in \{s\} \times V\} \cup \{((l, C_m), t) \in V \times \{t\}\}. \end{aligned}$$

So there is an edge from each literal occurrence in C_i to each literal occurrence in C_{i+1}, and edges from s to the literal occurrences of C_1, and edges from the literal occurrences of C_m to t. Let

$$S = \{((k, C), (l, D)) \in V^2 \mid \ k, l \text{ are the negations of each other}\}.$$

We claim that $F \in$ SAT iff $(G, s, t, S) \in L$. If $F \in$ SAT with satisfying assignment a, then define p to start at s,select a literal occurrence true at a from each successive clause, and then end at t. p does not violate any of the constraints in S since a is a consistent assignment.

Conversely, if $(G, s, t, S) \in L$, then let p be a satisfying path from s to t. Define variable assignment a as follows: if p passes through variable x, then assign x true, if p passes through $\neg x$, then assign x false, otherwise assign x arbitrarily. a is well-defined since the constraints in S preclude p from passing through both x and $\neg x$. Since for each clause C, p passes through at least 1 literal of C, a must satisfy C, and hence F. So $F \in$ SAT. $\square$