CSE 200 Homework 2

Chris Calabro

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Claim 1. PSPACE $\neq$ E.

Proof. Supposing not, we will show that EXP $\subseteq$ PSPACE $=$ E, which will contradict the time hierarchy theorem. (To see this, consider time constructible function $f(n) = 2^{n^2}$. By the time hierarchy theorem, there is a language L decidable in time $f(n)^{O(1)} = 2^{O(n^2)}$, but not in time $f(n) = 2^{n^2}$. In follows that L is in EXP because $2^{O(n^2)} \leq 2^{n^3}$, but not in E because $f(n) = 2^{n^2}$ is bigger that 2^{cn} for any constant c.) So let $A \in$ EXP. Then $\exists M_A \in \text{TM}, a > 0$ such that $\mathcal{L}(M_A) = A$ and M_A runs in time 2^{n^a}. We now use a padding argument: let # be a fresh symbol and define

$$B = \{x\#^{|x|^a - |x|} \mid x \in A\}.$$

Define

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$M_B(y)$
  if $y$ has the form $x\#^{|x|^a-|x|}$, return $M_A(x)$
  else reject
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Then $\mathcal{L}(M_B) = B$ and M_B runs in time polynomial in $2^{|x|^a} = 2^{|y|}$ (i.e., $2^{c|y|}$ for some constant c) and so $B \in \text{E} = \text{PSPACE}$. So $\exists N_B \in \text{TM}, b > 0$ such that $\mathcal{L}(N_B) = B$ and N_B uses space $\leq n^b$. Define

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$N_A(x)$
  return $N_B(x\#^{|x|^a-|x|})$
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Then $\mathcal{L}(N_A) = A$ and N_A uses space $\leq |x|^{ab}$. So $A \in$ PSPACE, which completes the proof. □

This argument is tricky because it uses the false hypothesis twice: once to construct N_B and again to contradict the time hierarchy theorem.

The intuition behind the proof is simply that the set of exponential functions of the form 2^{cn} is not closed under composition with polynomials on the right, while the set of polynomials is. So if PSPACE = E, then one should be able to replace a polynomial in the running time of a machine by a linear function by first converting to a space-bounded machine.