

CSE 200 Homework 2

Chris Calabro

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Claim 1. $\text{PSPACE} \neq \text{E}$.

Proof. Supposing not, we will show that $\text{EXP} \subseteq \text{PSPACE} = \text{E}$, which will contradict the time hierarchy theorem. (To see this, consider time constructible function $f(n) = 2^{n^2}$. By the time hierarchy theorem, there is a language L decidable in time $f(n)^{O(1)} = 2^{O(n^2)}$, but not in time $f(n) = 2^{n^2}$. It follows that L is in EXP because $2^{O(n^2)} \leq 2^{n^3}$, but not in E because $f(n) = 2^{n^2}$ is bigger than 2^{cn} for any constant c .) So let $A \in \text{EXP}$. Then $\exists M_A \in \text{TM}, a > 0$ such that $\mathcal{L}(M_A) = A$ and M_A runs in time 2^{n^a} . We now use a padding argument: let $\#$ be a fresh symbol and define

$$B = \{x\#^{|x|^a - |x|} \mid x \in A\}.$$

Define

$M_B(y)$
if y has the form $x\#^{|x|^a - |x|}$, return $M_A(x)$
else reject

Then $\mathcal{L}(M_B) = B$ and M_B runs in time polynomial in $2^{|x|^a} = 2^{|y|}$ (i.e., $2^{c|y|}$ for some constant c) and so $B \in \text{E} = \text{PSPACE}$. So $\exists N_B \in \text{TM}, b > 0$ such that $\mathcal{L}(N_B) = B$ and N_B uses space $\leq n^b$. Define

$N_A(x)$
return $N_B(x\#^{|x|^a - |x|})$

Then $\mathcal{L}(N_A) = A$ and N_A uses space $\leq |x|^{ab}$. So $A \in \text{PSPACE}$, which completes the proof. $\square$

This argument is tricky because it uses the false hypothesis twice: once to construct N_B and again to contradict the time hierarchy theorem.

The intuition behind the proof is simply that the set of exponential functions of the form 2^{cn} is not closed under composition with polynomials on the right, while the set of polynomials is. So if $\text{PSPACE} = \text{E}$, then one should be able to replace a polynomial in the running time of a machine by a linear function by first converting to a space-bounded machine.